MODULE* – 7

SRICTLY CONVEX LINER METRIC SPACES AND THEIR GENERALIZATION

INTRODUCTION

The present module is a study of strictly convex linear metric spaces. The notion of strict convexity in normed linear spaces was introduced independently by J.A. Clarkson and M.G. krein in 1935 ([5]). Different mathematicians calls strictly convex normed spaces by different names such as strictly normalized , rotund and strongly convex ([5]). Ahuja , Naran and Trehan ([1]) extended the notion of strict convexity to linear metric spaces in 1977 . Later the study was pursued by K.P.R. Sastry, S.V.R. Naidu, T.D. Narang, concept or strict convexity to linear metric spaces, whatever literature was available, the uniquencess of best approximation was established only in normed linear spaces and not in linear metric spaces. This motivated the extension of the notion of strict convexity to linear metric spaces. Various forms of strict convexity and its relation with other spaces have been discussed by sastry and naidu ([9]) and ([10]). Some characterizations of strictly convex linear metric spaces were given by Naran in [3] and [4]. Throughout this module , the underlying field will be either the field of real numbers or the field of complex numbers.

This module 'Strictly Convex Linear Metric Spaces and their Generalizations' has been divided into two sections. The first section deals with the definition and some examples of strictly convex linear metric spaces. In this section, we also prove some properties of linear metric spaces. In the second section, we give specail linear metric spaces i.e. linear metric spaces with properties : A, B, C, S.C., P.S.C., B.C., P and P_1 To star with, we set up some notations and terminology to be used in this Module 7 and Module 8.

The symbol ∈ will stand for belongs to, iff for if and only if, s.c. for strictly convex, n.1.s. for normed linear space, dim A for dimenstion of a set A, f.d. for finite dimensional, inf. for inrimum, sup. for supremum, min. for minimum, max. for maximum, int. for interior, R^n for the n-dimensional Euclidean space, C^n for n-dimensional unitary space, R^+ for the set of non-negative real numbers. ∅ for the empty set, conv (A) for the convex hull of a set A, Cl A for the closure of a set A, ∂A denotes the topological boundary of A, E/G for the quotient space of E by G, E\G for the complement of a set G in E, [x] for the subspace generated by an element X, R[f (x)] for the real part of f(x), Im[f(x)] for the imaginary part of f(x), d (x, G) for the distance of x from a set G, line segment [x, y] for the set $\{\alpha x + (1-\alpha)y, 0 \leq \alpha \leq 1\}$,] x, y [for the open line segment $\{\alpha x + (1-\alpha)y, 0 < \alpha < 1\}$, $x_n \rightharpoonup x$ for x_n converges to x weakly. In a linear metric space (X, d), $B[0, r] = \{x \in X, d(0, x) \leq r\}$ will stand for the closed ball with centre 0 and radius r, $B(0, r) = \{x \in X, d(0, x) < r\}$ will stand for the open ball with centre 0 and radius r and the functional $d(0,\cdot) = \|\cdot\|$ defined on X is called quasinorm of X .

Other notations will be given whenever these occur. The numbers within square barackets indicate references cited at the end of the Mod.

1. PRELIMANARIES

Definition 1.1 A subset A or a linear space X is said to be **convex** if with any two points x, y of A, it contains the line segment joining two points i.e. x, y ϵ A, $\alpha \in]\,0,1\,[$ imply αx + (1- α)y ϵ A. It is said to be **mid-point convex** if $\frac{1}{2}x + \frac{1}{2}y$ ϵ A for any two points x, y ϵ A.

Definition 1.2 If A is any set in a linear space X then intersection of all the convex sets containing A is called the **convex hull** of A.

Definition 1.3 A set V in a linear space X is said to be **linear manifold** if it is of form V = x_0 + G = { x_0 + g : g ϵ G}, where x_0 ϵ X and G is a linear subspace of X i.e. a translate of a linear subspace of X is called a **linear manifold.**

A closed linear manifold H⊂X is called a **hyperplane** if there exists no closed linear manifold $H_1 \subset X$ such that $H \subset H_1$ and H≠ X i.e. H is the maximal closed linear manifold in X.

Definition 1.4 A set of the form {x ϵ L, r (x)≥ ∝} where L is an n-dimensional subspace of a linear space X (n, a natural number, 1 ≤ n ≤ dim X ≤ ∞) f is a linear functional on L, and α is a real number, is called an (closed) **n-dimensional half plane.**

Definition 1.5 A subset A of a linear space X is called **symmetric** if {-x : x ϵ A} = A.

Definition 1.6 A set X with a family β of subsets of X is called a **topological space** if β satisfies the following conditions:

(a) The empty set ∅ and the whole set X belong to β.

(b) The union of any number or mebers of β is again a member of β.

(c) The intersection of any finite number of members of β is again a member of β.

The family β is called a **topology** for X, and the members of β are called open sets of X in this topology.

Definition 1.7 A vector space X over a field K, together with a Housdorff topology β is called a **topological linear space** if the vector space operations $(x, y) \to x + y$ from X x X into X and $(\alpha, x) \to \alpha x$ from K x X into X are continous.

Definition 1.8 A topological linear space is said to be **locally convex** if it has a base of convex neighbourhoods.

Definition 1.9 A linear space X is called a **linear metric space** if it is a topological linear space with topology derived from an invariant metric i.e. $d(x_1 + y, x_2 + y) = d(x_1, x_2)$ for every choice of x_1, x_2 and y in X.

Equivalently, a metric space (X, d) is said to be a linear metric space if

(i) It is a linear space.

(ii) Addition and scalar multiplications are continuous i.e. for $< x_n > \to x$, $< y_n > \to y$, $< \alpha_n > \to \alpha$,
$< x_n + y_n > \to x + y$ and $\alpha_n x_n > \to \alpha x$, and

(iii) d is translation invariant i.e. $d(x_1 + y, x_2 + y) = d(x_1, x_2)$ for every choice of x_1, x_2, y, in X.

Definition 1.10 A linear metric space (X, d) is said to be **bounded liner metric space** if the metric d is bounded i.e. there exists r > 0 such that

$$r = \sup_{x \in X} d(x, 0).$$

Definition 1.11 A linear metric space (X, d) is said to be **strongly locally convex** if each open sphere in it is a convex set.

Definition 1.12 A linear space X is said to be **normed linear space** if to each X ϵ X, there is assigned a unique real number, which we denote by $\|x\|$, satisfying the following properties:

(i) $\|x\| \geq 0$ and $\|x\| = 0$ iff $x = 0$

(ii) $\|x + y\| \leq \|x\| + \|y\|$

(iii) $\|\alpha\, x\| = \|\alpha\| + \|x\|$, y ϵ X and for all scalars α.

Definition 1.13 A normed linear space $(X, \|\cdot\|)$ is said to be **strictly convex** if for any two points x and y or X with $\|x\| = \|y\| = 1$, $\left\|\frac{x+y}{2}\right\| < 1$ unless x = y.

Definition 1.14 The **conjugate space** or **dual space** of a n.1.s. X, denoted by X^*, is the space of all continuous linear functionals on X with the usual linear space operations and the norm defined by

$$\|f\| = \sup_{\substack{\|x\| \leq 1 \\ x \in X}} |f(x)|$$

Definition 1.15 A normed linear space X is said to be **reflexive** if X^{**} = X, where X^{**} stands for the **second conjugate space** of X.

Definition 1.16 A n. 1. s. X is said to **be smooth** if every element x of X with $\|x\| = 1$ has a unique support hyperplane to the open unit ball B(0, 1) = $\{x : x \in E, \|x\| < 1\}$.

Definition 1.17 A normed linear space X is said to be **strictly normed** if the relation $\|x + y\| = \|x\| + \|y\|$, x, y $\in$ $x \backslash \{0\}$ implies the existence of a number C > 0 such that y = C x Such a norm is called a **strict norm.**

Definition 1.18 An element x of a n.1,s. X is said to be **orthogonal** to an element y of X, $x \perp y$, if $\|x+\alpha y\| \geq \|x\|$ for every scalar α. x is said to be **orthogonal to a subset** G of X, $x \perp G$, if $x \perp y$ for all $y \in G$.

Definition 1.19 Let X be a linear space. A mapping

$$<\star, \star> \quad : X \times X \longrightarrow k,$$

the field or scalars is said to be an **inner product** on X if the following properties hold for all x, y, z ∈ X and for all scalars α, β ∈ K

(i) $< x, x > \geq 0$ for all $x \in X$

(ii) $< x, y > = \overline{< y, x >}$ where $\overline{< y, x >}$ denotes the complex conjugate of $< y, x >$

(iii) $< \alpha x + \beta y, z > = \alpha <x, z> + \beta <y, z>$.

The pair $(X , <\cdot, \cdot>)$ is called an **inner product space.**

Definition 1.20 A complete inner product space is called a **Hilbert space.**

Definition 1.21 Let $(X, \Im)$ and (Y, β) be two topological spaces. Then a mapping $f : X \to Y$ is called a **homeomorphism** if

(i) f is one-one
(ii) f is onto
(iii) f is continous
(iv) f^{-1} is continous

Definition 1.22 If E and F are two normed liner spaces over the same field K then a mapping T : E ⟶ F is called a **linear transformation** if it satisfies the following properties:

(i) $T(x + y) = Tx + Ty$

(ii) $T(\alpha x) = \alpha T(x)$ for any $x, y \in E$ and for all $\alpha \in K$.

Linear transformation T is said to be bounded if

$$\|T\| = \sup_{\substack{\|x\| \le 1 \\ x \in E}} \|T(x)\| \text{ is finite}$$

<u>Definition</u> 1.23 A subset A of a metric space (X, d) is said to be **<u>metrically bounded</u>** or **<u>d-bounded</u>** if sup {d (x, y), x, y$\in$ A } is finite.

<u>Definition</u> 1.24 Let G be a subset of a metric space (X, d) and x be an element of X. An element $g_0 \in$ G is called a **<u>best</u>**-approximation or a **<u>nearest point</u>** to x in G if

$$d(x, g_0) = d(x, G) = \inf_{g \in G} d(x, g)$$

The set of all elements $g_0 \in$ G is denoted by $L_G(x)$

i.e. $L_G(x) = \{ g_0 \in G, d(x, g_0) = d(x, G) \}$

G is said to be **<u>proximinal</u>** if $L_G(x)$ is non-empty for every x $\in$ X. G is said to be **<u>semi-Chebyshev</u>** if $L_G(x)$ is atmost singleton for every x $\in$ X and G is aid to be **<u>Chebyshev</u>** or **<u>uiquely proximinal</u>** if $L_G(x)$ is exactly singleton for every x $\in$ X .

.

2.1 <u>SOME BASIC PROPERTIES OF STRICTLY CONVEX LINEAR METRIC SPACES</u>

We begin with the notion of strictly convex linear metric space as introduced by Ahuja, Narang and Trehan in [1].

Definition 2.1.1 [1] A linear metric space (X, d) is said to be **strictly convex** if $d(x,0) \le r, d(y,0) \le r$ imply d ($\frac{x+y}{2}$, 0) < r unless x = y, x, y$\in$ X and r is any positive real number.

Next we give an example of a strictly convex linear metric space.

Example 2.1.1 [1] The set R of real numbers with metric d defined by d (x, y) = $\frac{|x-y|}{1+|x-y|}$ is a strictly convex linear metric space. This can be seen as follows:

Lex x and y be two distinct points of R with d (x, 0) ≤ r, d (y, 0) ≤ r. These give

$$|x| \le \frac{r}{1-r}, \quad |y| \le \frac{r}{1-r}$$

Strict convexity of |·| implies $\left|\frac{x+y}{2}\right| < \frac{r}{1-r}$, which in turn implies

$$d\left(\frac{x+y}{2}, 0\right) < r$$

Remark 2.1.1 [1] In fact we can say a little more viz. if (X,‖·‖) is a strictly convex normed linear space then the linear metric space (X, d) where d is defined as $d(x,y) = \frac{\|x - y\|}{1+\|x - y\|}$ is strictly convex.

The following example shows that even a finite dmensional liner metric space need not be strictly convex.

Example 2.1.1 [1] Consider (R^2, d) where d is defined as

$$d(x,y) = \max\{|x_1-y_1|, |x_2-y_2|\}$$
$$x = (x_1, x_2), \; y = (y_1,y_2).$$

Let x = (1, 1) and y = (1, 0) then d (x , 0) = 1, d (y, 0) = 1 and also d $\left(\frac{x+y}{2}, 0\right) = 1$. Therefore (R^2, d) is not strictly convex.

The following theorem tells that certain type of linear metric spaces can never be strictly convex.

Theorem 2.1.1 [8] A non-zero bounded linear metric space in which the metric attains its superemum is not strictly convex.

Proof : Let (X,d) be a non-zero bounded linear metric space such that d attains its superemum, say r. Then r > 0 and there exists z $\in$ X such that d (z, 0) = r. Take x = $\frac{z}{2}$ and y = $\frac{3}{2}$ z. Then d (x, 0) $\leq$ r, d (y, 0) $\leq$ r and x $\neq$ y, but d($\left(\frac{x+y}{2}, 0\right)$=d (z, 0) = r . Hence (X, d) is not strictly convex.

It is well known (see [6]) that Every Convex Proximinal set in a Strictly Convex Normed Linear Space is Chebyshev. We show below that a similar result holds in a strictly convex linear metric space.

Theorem 2.1.2 [1] A convex proximinal set in a strictly convex linear metric space is Chebyshev.

Proof : Let G be a convex proximinal set in a strictly convex linear metric space (X, d) and let p be any arbitary point on X. Since G is proximinal, there exists $g_1^* \in G$ such that d (p, g_1^*) = d (p, G) $\equiv$ r (say).

Let, if possible, there exists $g_2^* \in G$ such that d (p, g_2^*) = r. Invariance of the metric d implies that

$$d(p - g_1^*, 0) = d(p - g_2^*, 0) = r.$$

Since X is strictly convex, we have

$$d\left(\frac{(p - g_1^*) + (p - g_2^*)}{2}, 0\right) < r \text{ unless } p - g_1^* = p - g_2^*$$

i.e. $d\left(p, \frac{g_1^* + g_2^*}{2}\right) < r$ unless $g_1^* = g_2^*$. Since $\frac{g_1^* + g_2^*}{2} \in G$, definition of r implies that $g_1^* = g_2^*$. Hence G is Chebyshev.

Now, we give a lemma to be used in Theorem 2.1.3 which shows that strictly convex linear metric spaces are strongly locally convex – a notion introcduced by T.D. Narang in [2].

Lemma 2.1.1 [8] Let (X, T) be a topological vector space and S be a non-empty closed subset of X such that x, y ϵ ∂S (boundry of S) and x ≠ y imply (x, y) ∩ S=∅. Then S is conve.

Proof: Suppose S is not convex. Then there exist x, y ∈ S, x ≠ y such that (x, y) ∩ S'=∅ (S' is the compement of S in X).

Let A = { t∈ (0, 1) : tx + (1-t) y ∈ S'} Then A is anon-empty subset of R. Let B be a component of A. Then there exist α, β ∈ R such that α < β = (α, β). write

$$z_1 = \alpha x + (1-\alpha)\, y \text{ and } z_2 = \beta\, x + (1-\beta)\, y$$

Then, clearly z_1, z_2 are distinct points of ∂S- and (z_1, z_2) ∩ S =∅ which contrdicts the hypothesis.

Theorem 2.1.3 [8] In a strictly convex linear metric space, the balls are convex.

Proof From lemma 2.1.1, it is clear that closed balls with centre at the origin and hence the open balls with centre at the origin are convex. Since every ball is a translate of a ball with centre at the origin, the result is immediate.

2.2 SOME SPECIAL LINEAR METRIC SPACES

In this section, we whall discuss some special linear metric spaces i.e. linear metric spaces with properties A, B, C, S.C., P.S.C., B.C., P and P_1 and the relationships of A, B, C, P.S.C., B.C., P and P_1 with S.C.

We say that a linear metric space (X, d) has the

Property :-

A : Given r > 0, ε > 0 there exists δ> 0 such that

B [0, r + δ] ⊂ B [0, r] + B [0, ε]

B : Given r > 0, ε > 0 there exists δ> 0 such that

d (x, 0) > r-δ $\Rightarrow$ sup {d (x +z, 0) : d (z, 0) <ε} > r.

C: Give r > 0, ε > 0 there exists δ > 0 such that r < d (x, 0) < r +δ ⇒ there exists y, z such that d (y, 0) = r, d (z, 0) < ε and x = y + z.

S.C. r > 0 , x ≠ y, d (x, 0) ≤ r $\Rightarrow$ $d(\frac{x+y}{2}, 0) < r$.

P.S.C. $x \neq 0, y \neq 0$, d (x + y, 0) = d (x, 0) + d (y, 0) $\Rightarrow$ y= tx for some t > 0.

P. A linear metric space (X, d) is said to have property (P) if the nearest point mapping shrinks distances whenever it exists.

B.C. r ≥ 0, d (x, 0) = d (y, 0) = r $\Rightarrow$ $d(\frac{x+y}{2}, 0) \leq r$.

P_1 A linear metric space (X, d) is said to have property (P_1) if for every pair of elements x, z ∈ X such that d (x + z, 0) ≤ d (x, 0) there exist constants b = b (x, z) > 0, C = C (x, z) > G such that d (y + C z, 0) ≤ d (y, 0) for d (y , x) ≤ b.

Lemma 2.2.1 [9] Let f : $R^+ \rightarrow R^+$ be strictly increasing function such that (X, $f \circ d$) is a linear meric space. Then (X, d) has S.C. ⇔ (X, $f \circ d$) has S.C.

Proof Let r > 0 and ($f \circ d$) (x, 0) ≤ r, ($f \circ d$) (y, 0) ≤ r i.e. f [d (x, 0)] $\leq$ r, f [d (y, 0)]$\leq$ r.

We may assume that there exists z$\in$X such that f [d (z, 0)]$\geq$ r. Since f (d(t z, 0)) is a continuous function of t on R and hence for some t $\in$] 0, 1 [, f (d (tz, 0)) = r so that $f^{-1}(r)$ exists.

Clearly $f^{-1}(r) > 0$

d (x, 0) $\leq$ f^{-1} (r), d (y, 0) $\leq$ f^{-1} (r)

and so by strict convexity of d, $d(\frac{x+y}{2}, 0) < f^{-1}$ (r) $\Rightarrow$

f $[d(\frac{x+y}{2}, 0)] < r$ as f is strictly increasing

Since f^{-1} is strictly increasing and d = $f^{-1} \circ (f \circ d)$, the other implication follows from the first.

As above, the following lemma can be easily established.

Lemma 2.2.2 [9] Let f $: R^{+} \rightarrow R^{+}$ be a strictly increasing function such that

(i) f (s + t) $\leq$ f (s) + f (t) for all s, t $\in$ R^{+} and

(ii) (X, $f \circ d$) is a linear metric space. Then (X, d) has P.S.C. ==> (X, $f \circ d$) has P.S.C.

Proof : Let ($f \circ d$) (x + y, 0) = ($f \circ d$) (x, 0) + ($f \circ d$) (y, 0)

i.e. f [d (x + y, 0)] = f [d (x, 0)] + f d (y, 0)]

Let d (x, 0) = s and d (y, 0) =t

Now f (s + t) $\leq$ r (s) + f (t)

$$\begin{aligned}
\Rightarrow \quad f\,(d\,(x, 0) + d\,(y, 0)) &\leq f\,(d(x, 0)) + f\,(d\,(y, 0)) \\
&= (f \circ d)\,(x + y, 0) \\
&= f\,(d\,(x + y, 0)) \\
&\leq f\,(d\,(x, 0) + d\,(y, 0)\,) \\
f\,(d\,(x, 0) + d\,(y, 0)) &= f\,(d\,(x + y, 0)\,)
\end{aligned}$$

Now since r is a strictly increasing function, we have

d (x, 0) + d (y, 0) = d x + y, 0)

which implies that y = tx for some t > 0 and so (X, $f \circ d$) has P.S.C.

Suppose f : $R^+ \to R^+$ is such that (X, $f \circ d$) is a linear metric space. Then f does not satisfy that condition f (s + t) < f (s) + f (t) for all s, t ∈ R^+ as is evident from the following example:

Example 2.2 1 [9] The function d defined by d (x, y) = $|x\text{-}y|^{1/2}$ is a linear lmeric on R. If we define f : $R^+ \to R^+$ by f (t) = t^2, then $f \circ d$ is the usual metric on R. Clearly f does not satisfy the condition.

f (s + t) ≤ f (s) + f (t) for all s, t ∈ R^+

Given two linear metrics on alinear space X, their Euclidean combination on X x X is a linear metric while the Euclidean combination of two strictly convex norms on X is strictly convex. On X x X, the same need not be true in the case of linear metrics as the following example shows.

<u>Example</u> 2.2.2. [9] Consider the strictly convex linear metric space (R, d_1), where d_1(s. t) = $|s\text{-}t|^{1/2}$ for all s, t ∈R. Then

$$d((x_1, y_1), (x_2, y_2)) = [|x_1 - x_2| + |y_1 - y_2|]^{1/2}$$

is the Euclidean combination of d_1 with itself and is a linear metric on R^2. Clearly d ((1, 0) , (0, 0)) = d ((0, 1), (0, 0)) = 1

and d (($\frac{1}{2}$, $\frac{1}{2}$), (0, 0)) = 1.

Hence (R^2, d) is not strictly convex.

Each of the following two examples shows that if (X, d) has P.S.C. then it need not have S.C. In the first example the balls are convex whereas in the second example all the balls are not convex.

<u>Example</u> 2.2.3 [9] Let f : $R^+ \to R^+$ be defined by

$$f(t) = \begin{Bmatrix} t & if & 0 \le t \le 1 \\ 1 & if & t > 1 \end{Bmatrix}$$

and d be the linear metric on R defined by d (0, t) = f ($|t|$) for all $t \in R$. Then (R, d) has (B.C.) and P.S.C. but not S.C.

<u>Example</u> 2.2.4 [9] Let f : $R^+ \to R^+$ defined by

$$f(t) = \begin{cases} t & \text{if} \quad 0 \le t \le 1 \\ \frac{1}{2}(1+\frac{1}{t}) & \text{if} \quad t \ge 1 \end{cases}$$

and d be the linear metric on R defined by d (0, t) = f ($|t|$) for all $t \in R$. Then (R, d) has P.S.C. but neither (B.C.) nor (A) nor (B).

Now we give two more examples giving the relation between S.C., (A) and (B).

<u>Example</u> 2.2.5 [8] Define f : $R^+ \to R^+$ defined by

$$f(t) = \begin{cases} t, & \text{if} \quad 0 \le t \le 1 \\ 1, & \text{if} \quad 1 < t < 2 \\ \frac{t}{2}, & \text{if} \quad t \ge 2 \end{cases}$$

and d: R x R $\to R^+$ by

d (x, y) = f ($|x - y|$).

Then (R, d) is a totally complete Linear metric space such that all of its balls are convex but it is not strictly convex. Further it satisfies (A) but not (B) even though d is unbounded.

<u>Example</u> 2.2.6 [8] Define f : $R^+ \to R^+$ by

$$f(t) = \frac{t}{1+t}$$

and $\quad$ d: R x R $\rightarrow R^{+}$

by $\quad$ d(x, y) = f (|x – y|).

Then (R, d) is a bounded strictly convex linear metric space satisfying (A) but not (B).

Next, we show that a totally complete linear metric space satisfies (A) and totally complete linear metric space, in presence of S.C., satisfies (B). This is the essence of our next theorem.

Theorem 2.2.1 [8] The following hold:

(i) A totally complete linear metric space satisfies (A).

(ii) A totally complete strictly convex linear metric space satisfies (B).

Proof (i) Let (X, d) be a totally complete linear metric space. Let r> 0 and ε > 0 suppose there does not exist δ > 0

such that

$$B[0, r+\delta] \subset B[0, r] + B[0, \varepsilon].$$

Then there exists $z_n \in X$ such that

$$d(z_n,\ 0) > r + \frac{1}{n} \text{ and}$$

$$z_n \notin B[0, r] + B[0, \varepsilon]$$

so that

$$d(z_n, 0) > r$$

Since (X, d) is totally complete, there exists a convergent subsequence $\{Z_{n_i}\}$ of $\{z_n\}$ with limit, say, z. Then d (z, 0) = r and hence z is an interior point of

$$B[0,r]+B[0,\varepsilon]$$

so that it contains infinitely many z_n which is a contradiction.

(ii) Let (X, d) be a totally complete, strictly convex linear metric space. Let r > 0 and ε >0.

Suppose there does not exist $\delta > 0$ such that d (x, 0) > r - δ implies

sup {d (x + z, 0) , d z, 0) < ε } > r.

Then there exists x_n $\in$ X Such that

$$d(x_n,0) > r - \frac{1}{n}$$

and

sup $\{d\ (x_n + z, 0) : d(z,0) < \varepsilon\} \leq r.$

Since (X, d) is totally complete, there exists a convergent subsequence $\{x_{n_i}\}$ of $\{x_n\}$ with limit, say, x. Then d(x, 0) =r and

$$\sup\{d(x + z, 0) : d\ (z, 0) < \varepsilon\} \leq r$$

Choose t> 0 such that d (t x , 0) < ε. Then

d ((1 + t) x, 0) ≤ r.

But, since (X, d) is strictly convex,

$r = d\ (x, 0) < d\ ((1 + t)\ x, 0)$

(from corollary 3.1 of the third chapter), which is a contradiction.

Then following theorem gives the structure of line segments in strictly convex linear metric spaces.

Theorem 2.2.2 [9] Let (X, d) be strictly convex and $r > 0$. Suppose $S\ [0, r] \neq \emptyset$ and y, z are distinct points of $B\ [\ 0, r\ ]$. Then

$$E = \{t \in R : ty + (1-t)z \in B[0,r]\}$$

is a compact convex subset of R.

Proof The convexity of E follows from that of B [0, r]

Clearly E is closed. Let v = (y – z). Suppose E is not bounded above. Then $[0,\infty) \subset E$ so that

$$z + t\ v \in B[0, r] \text{ for all } t \in R^{+}$$

for any, $s \in (0, 1)$ and $t \in R^{+}$ we have

$$sz + tv = s(z + \frac{t}{s}v) + (1-s)0 \in B[0,r]$$

Since B[0, r] is convex, $tv \in B[\ o, r]$ for any $t \in R^{+}$ and hence for any $t \in R$. Let $x \in S[0,r]$ (such a point exists by hypothesis). For any $x \in (0, 1)$ and $t \in R$ we have

$$sx + tv = sx + (1-s)\ (\frac{t}{1-s})v \in B\ [0, r]$$

Hence $x + tv \in B\ [0,r]$ for all $t \in R$.

In particular

$$x + v,\ \ x - v \in B\,[\,0, r]$$

Also

$$x + v \neq x - v$$

and $x = \frac{1}{2}(x + v) + \frac{1}{2}(x - v)$

so from the strict convexity $x \in B\ (0, r)$, which is a contradiction. Therefore E is bounded above. Similarly it can be shown that E is bounded below. This completes the proof.

Note: Example 2.2.3 shows that the above result need not be valid if 'strict convexity' is replaced by ball convexity.

Corollary 2.2.1 [9] Let (X , d) be strictly convex and r > 0. Suppose $x \in X$, S [0, r] $\neq \emptyset$ and y, z are distinct points of B[x, r]. Then

$$E = \{t \in R : ty + (1 - t)z \in B[x, r]\}$$

is a compact convex set.

Proof Since we can write E as

$$E = \{t \in R : t(y - x) + (1 - t)\ (z - x) \in B[0, r]\},$$

the result follows from Theorem. 2.2.1

Corollary 2.2.2 [9] Let (x, d) be a strictly convex linear metric space. Then sup {d (t x, 0): t$\in$R} is invariant on X\{0}. In fact,

$$\sup\{d\,(t\,x, z) : t \in R\}$$

is invariant on $(x\backslash\{0\}) \times X$.

Proof: Let u, $v \in X$ and x, $y \in X\backslash\{0\}$ let

$r = \sup \{d(tx, u) : t \in R\}$

and $s = \sup \{d(ty, v) : t \in R\}$

Suppose r < s then there exists $a \in R$ such that

$d(\alpha y, v) = r$

Hence $S[0, r] \neq \emptyset$.Also 0 and x are distinct points of B [u, r] and

$\{t \in R : tx + (1-t)\, 0 \in B[u, r]\} = R$

which is false in view of corollary 2.2.1. Hence $r \nless s$ similarly, it can be shown $s \nless r$. Here r = s.

In Corollary 2.2.2, we have shown that in the presence of strict convexity,

$\sup \{d(tx, 0) : t \in R\}$

is invariant on $X\backslash\{0\}$.

In a strictly convex linear metric space, every half-ray emanating from the centre of a ball, passes through its surface, provided, of course, the surface is non-empty. This is the essence of

<u>Corollary</u> 2.2.3 [9] Let (x, d) be strictly convex, r > 0 and $s[0,r] \neq \emptyset$. Suppose x, $y \in x$ and $y \neq 0$. Then

$x + \alpha y \in S[x, r]$

for some $\alpha \in R^{+}$

Proof Let z ∈ S[0,r]. Then, from corollary 3.1 (Chapter III) follows that

sup {d(tz, 0)/ t∈R} > r.

Hence, by Corollary 2.2.3,

sup {d(ty, 0)/ t∈R} > r.

Consequently,

$\exists s \ \ \alpha \in R^{+}$

such that α y ∈S [0, r]

so that

x + α y ∈S [x, r].

The above two results give the impression that strictly convex liear metric spaces behve like normed linear spaces.

The following two examples show that mere ball convexity does not guarantee the invariance of

sup {d (tx, 0) : t∈ R}   on X\ {0}

In the first example we follow the technique used in the proof of the following result of Walter Rudin (Theorem 1.24 of [7]). "If X is a topological vector space with a countable local base, then there is a metric d on X such that

(a) d is compatiable with the topology of X

(b) The open balls centred at 0 are balanced, and

(c) d is invariant i.e

$$d(x + z, y + z) = d(x, y) \text{ for } x, y, z \in X.$$

If, in addition, X is locally convex, then d can be choosen so as to satisfy (a), (b), (c) and also

(d) all open balls are convex."

Example 2.2.7 [9]

Let $V_1 = \{(x, y) \in R^2 : |y| < \frac{1}{2}\}$, and

$$V_n = \{(x, y) \in R^2 : (x^2 + y^2)^{1/2} < \frac{1}{2^n}\} \text{ for } n = 2, 3 \ldots$$

Then

$$\{V_n : n = 1, 2, \ldots\}$$

is a balanced convex local base at the origin for the Euclidean topology on R^2. Also

$$V_{n+1} + V_{n+1} \subset V_n \; for \;\; n = 1, 2, \ldots$$

Let D be the set of rational numbers of the form

$$r = \sum_{n=1}^{\infty} C_n(r) \frac{1}{2^n}$$

where each of the digits $C_i(r)$ is 0 or 1 and only finitely many are 1. Define

$$A(r) = \begin{cases} R^2 & if \quad r \geq 1 \\ \sum_{n=1}^{\infty} C_n(r) V_n & if \quad r \in D. \end{cases}$$

Also define

$f(x) = \inf\{n : x \in A(r)\}$ for $x \in R^2$

and $d(x, y) = r(x - y)$, $(x \in R^2, y \in R^2)$

Then (R^2, d) is a linear metric space, all of whose balls are convex. Further

$$\sup\{d(t(0, 1), (0, 0)) : t \in R\} = \frac{1}{2}$$

and

$$\sup\{d(t(1, 0), (0, 0)) : t \in R\} = 1$$

The metric d in this example can be explicitly expressed as follows:

Let $(x, y) \in R^2$. Then

$$d(x, y), (0, 0)) = \begin{cases} \|(x,y)\| & \text{if} \quad \|(x,y)\| \le \frac{1}{2} \\ \frac{1}{2} & \text{if} \quad \|(x,y)\| \ge \frac{1}{2} \text{ and } \|y\| \le \frac{1}{2} \\ |y| & \text{if} \quad \frac{1}{2} \le |y| \le 1 \\ 1 & \text{if} \quad |y| \ge 1 \end{cases},$$

where $\|(x, y)\| = (x^2 + y^2)^{1/2}$

is the Euclidean norm. Here we have

$$S[0,r] = \begin{cases} \{(x,y) \in R^2 : \ \|x,y\| = r\} \text{ if } 0<r<\frac{1}{2} \\ \{(x,y) \in R^2 : \ \|x,y\| \ge \frac{1}{2}, \|y\| \le \frac{1}{2}\} \text{ if } r = \frac{1}{2} \\ \{(x,y) \in R^2 : \ |y| = r\} \quad \text{if } \frac{1}{2} < r < 1 \\ \{(x,y) \in R^2 : \ |y| \ge 1\} \quad \text{if } \frac{1}{2} < r = 1 \end{cases}$$

Example 2.2.8 [9] Define d on R^2 as follows:

$$
d((x,y),(0,0)) \begin{cases} \|(x,y)\| & \text{if } \|(x,y)\| \le \frac{1}{2} \\ \frac{1}{2} & \text{if } \|(x,y)\| \ge \frac{1}{2} \text{ and } |y| < \frac{1}{2} \\ |y| & \text{if } |y| \ge \frac{1}{2} \end{cases}
$$

where

$$\|(x, y)\| = (x^2 + y^2)^{1/2}$$

is the Euclidean norm. Then (R^2, d) is a linear metric space with ball convexity. The metric nature of d follows immediately, if we observe that

$$|y| \le d((x, y), (0, 0) \le \|x, y\|$$

for all (x, y) $\in R^2$

In example 2.2.4, the '**<u>superemum</u>**' is finite in each direction whereas, here the '**<u>superemum</u>**' is finite in one direction and infinite in another. Infact,

$$\sup\{d((t, 0), (0, 0)) : t \in R\} = \frac{1}{2}$$

and

$$\sup\{d((0, t), (0, 0) : t \in R\} = \infty$$

Now we shall show that closed balls in a strictly convex linear metric space with non-empty surface are compact if the space is finite dimensional. We shall be using the following result of Walter Rudin (Theorem 1.28 (b) [ii]. "If $\{x_n\}$ is a sequence in a metrizable topological vector space X and if $x_n \to 0$ as $n \to \infty$, then there are positive scalars γ_n such that

$$\gamma_n \to \infty \text{ and } \gamma_n x_n \to 0$$

Theorem 2.2.3 [9] Let (X, d) be strictly convex linear metric space and finite dimensional. If S {0, r] ≠ Ø then B [0, r] is compact.

Proof Since (X, d) is finite dimensional linear metric space, it is normable . Let $\|\cdot\|$ be a norm compatible with the topology on X. Since B [o,r] is closed, it is sufficient to show that it is norm bounded. Suppose that B [0, r] is unbounded in the norm. Then there exists a sequence $\{t_n\}$ of positive scalars and a sequence $\{x_n\}$ of vectors of unit norm such that

$t_n \to \infty$ as $n \to \infty$ and $t_n x_n \in$ B [0, r) for all n. Since (X $\|\cdot\|$) is a finite dimensional normed linear space, $\{x_n\}$ has a convergent subsequence. We may suppose that $\{x_n\}$ is convergent with limit, say x_0. Then $(\|x_0\|)=1$. Since the norm topology and d-topology are the same,

$$d(x_n - x_0, 0) \to 0 \text{ as } n \to \infty$$

Hence by the above theorem, there exists a sequence $\{\alpha_n\}$ of positive scalars such that

$$\alpha_n \to \infty \text{ and } d(\alpha_n(x_n - x_0), 0) \to 0 \text{ as } n \to \infty$$

Let $\beta_n = \min\{t_n, \alpha_n\}$. Then

$$\beta_n \to \infty \text{ as } n \to \infty.$$

For a positive integer m let ε > 0. There exists n > m such that

$\beta_n > \beta_m$ and d $(\beta_n (x_n - x_0), 0) < \varepsilon$.

Now

$$d(\beta_m x_0, 0) \leq d(\beta_n\ x_0,\ 0)$$
$$\leq d(\beta_n(x_n - x_0), 0) + d(\beta_n\ x_n,\ 0)$$
$$\leq \varepsilon + d(t_n\ x_n,\ 0)$$
$$\leq \varepsilon + r.$$

This being true for each ε > 0, we have

$$d(\beta_m\ x_0, 0) \leq r$$

so that $\beta_m x_0 \in B[0,r]$ for each m. Since

$\beta_n \to \infty$ and B [0, r] is convex, it follows that

(1) $\sup\{d(t x_0, 0) : t \in R\} \leq r$

But, since S[0,r] ≠ ∅, there exists y ϵ S[0,r] and by theorem 3.1, it follows that

(2) $\sup\{d(t y, 0) : t \in R\} > r$

(1) and (2) contradict each other in view of corollary 2.2.2.

Hence B [0, r] is compact.

Next theorem shows that a strictly convex finite dimensional linear metric space in the presence of an unbounded metric is totally complete, a notion introduced in [1]

<u>Lemma</u> 2.2.3 [9] A strictly convex finite dimensional linear metric space with an unbounded metric is totally complete.

<u>Proof</u> : Let (X, d) be strictly convex, finite dimensional linear metric space and d be unbounded. Let r > 0. Since d is unbounded, there exists y ϵ X such that

$$d(y, 0) > r.$$

Hence by the continuity, there exists tϵ] 0, 1 [such that d (t y, 0) = r so that S [0, r] ≠ ∅. Therefore by Theorem 2.2.3, B [0, r] is compact. Hence, every closed ball and therefore every d-bounded closed set is compact.

In a linear metric space if the metric is additive along a half-ray emanating from the origin, then it is a norm along the line determined by the half-ray. More generally, we have the following result, the proof of which is immediate.

<u>Lemma</u> 2.2.4 [9] Suppose x_0 , y_0 ϵ X are such that

(1) $d(x, y) = d(x, z) + d(z, y) \ \forall \in [x, y]$

whenever x, yϵ $[x_0, y_0]$. Then

$$d(t x_0, t y_0) = t\, d(x_0, y_0) \text{ for all } t \in [0, 1].$$

The above result need not be true, even when (X, d) is strictly convex if (1) is replaced by

$$d(x_0, y_0) = d(x_0, z) + d(z, y_0) \quad \forall z \in (x_0, y_0)$$

as the following example shows:

<u>Example</u> 2.2.9 {15] Define f : $R^+ \to R^+$ by

$$f(t) = \begin{cases} \frac{4}{3}t & \text{if} \quad 0 \le t \le \frac{1}{4} \\ \frac{2}{3}t + \frac{1}{6} & \text{if} \quad \frac{1}{4} \le t \le \frac{3}{4} \\ \frac{2}{3}t + \frac{1}{3} & \text{if} \quad t \ge 1 \end{cases}$$ Then (R, d), where

$$d(x, y) = f(|x-y|) \; \forall \, x, y \in R,$$

is a strictly convex linear metric space. Clearly

$$d(0, 1) = d(0, t) + d(t, 1) \; \forall \, t \in [0, 1]$$

but

$$d(0, t) = td(0, 1) \; \forall \, t \in [0, 1]$$

is not true.

The following example shows that the distance between two points can be the sum of their distances from an intermediate point but at the same time it may not be so for every intermediate point, even in a strictly convex linear metric space.

<u>Example</u> **2.2.10 [9]** Define f : $R^+ \to R^+$ by

$$f(t) = \begin{cases} \frac{2t}{1+t} & if \quad 0 \le t \le 1 \\ t & if \quad t \ge 1 \end{cases}$$

Then (R, d), where

$$d(x, y) = f(|x-y| \; \forall \; x, y \in R,$$

is a strictly convex linear metric space. We have

$$d(3, 4) + d(4, 5) = d(3, 5) \ne d(3, \tfrac{7}{2}) + d(\tfrac{7}{2}, 5)$$

Now we show that strict convexity is weaker than property (P) but stronger than the property (P_1). This is the essence of our next theorem.

<u>Theorem</u> 2.2.4 [4] Let (X, d) be a linear metric space, we have:

(i) If (X, d) has property (P) then it is strictly convex.

(ii) If (X, d) is strictly convex then it has property (P_1).

<u>Proof</u> (1) Suppose (X, d) is not strictly convex. Then by lemma 3.2 ([3]) there exists an r > 0 and distinct points x and y such that

$$d(x, 0) = d(y, 0) = r$$

and $B[0, r] \cap]x, y[= \emptyset$.

consider the compact line segment [x, y]. This set is proximinal let $f : E \rightarrow [x, y]$ be the nearest point mapping then

$f(0) = x$, $f(0) = y$. Consider

$$d(x, y) = d(f(0), f(0)) \leq d(0, 0) = 0$$

[By property (P)] and so x = y, a contradiction.

(ii) If $d(x + z, 0) < d(x, 0)$ and

$$2d(y, x) \leq d(x, 0) - d(x + z, 0)$$

then

$$d(y + z, 0) \leq d(x + z, 0) + d(y, x) \leq d(y, 0)$$

Thus property (P_1) is satisfied if

$$b = [d((x,0) - d(x + z, 0)]/2 \text{ and } C = 1.$$

If d (x + z, 0) = d x, 0)

then by the strict convexity,

$$d\left(x + \frac{z}{2}, 0\right) = d\left(\frac{x + z + x}{2}, 0\right) < d(x, 0)$$

and so property (P_1) is satisfied if

$$b = [d\{x, 0) = d(x + \frac{z}{2}, 0)]/2 \text{ and } C = \frac{1}{2} \text{ as}$$

$$d\left(y + \frac{z}{2}, 0\right) \leq d(y, x) + d\left(x + \frac{z}{2}, 0\right)$$

$$= d(y, x) + d(x, 0) - 2b$$

$$\leq x(x, 0) - b$$

$$\leq d(y, 0).$$

Theorem 2.2.5 [1] A complete convex set K in a linear metric space (X, d) satisfying the property (P) is Chebyshev.

Proof : Let g ϵ X and

$$r = \inf\{d(x, g) : x \in K\}$$

By definition of infimum there is a sequence $< x_n >$ in K such that

$$\lim_{n\to\infty} d(x_n, g) = \inf\{d(x, g) : x \in K\}$$

By property (P) we have $< x_{n_k} >$ in K. K being complete,

$$< x_{n_k} > \longrightarrow x \in K$$

and consequently

$$d(x^*, g) \geq r.$$

Also $d(x^*, g) \leq d(x^*, x_{n_k}) + d(x_{n_k}, g)$

implies

$$d(x^*, g) \leq r.$$

Hence

$$d(x^*, g) = r$$

Now, if possible

$$x_1^*, x_2^*, \in K$$

be such that

$$d(x_1^*, g) = d(x_2^*, g) = r.$$

Consider the sequence $< x_n >$ defined as

$$x_n = \begin{cases} x_1^* & \text{if } n \text{ is odd} \\ x_2^* & \text{if } n \text{ is even.} \end{cases}$$

Then $\lim_{n\to\infty} d(x_n, g) = d(x_1^*, g) = d(x_2^*, g) = r \equiv \inf\{d(x, g) : x \in K\}$.

By property (P), $< x_n >$ has a Cauchy sequence $< x_{n_k} >$ and therefore for a given ε > 0, there exists a positive

integer N such that

$$d(x_{n_k}, x_{m_k}) < \varepsilon \text{ for all } n_k, m_k \geq N,$$

$$\text{i.e. } d(x_1^*, x_2^*) < \varepsilon$$

ε being arbitrary, $x_1^* = x_2^*$.

REFERENCES

[1] G.C. Ahija, T.D. Narang and Swaran Trehan: Best approxion conves sets in metric linear spaces, Math, Nachr. 78(1977) , 125-130

[2] T.D. Narang: on certain Characterizations of Best Approximation in metric linear spaces, Pure and Applied Mathematika Sciences 4 (1976), 121-124.

[3] T.D. Narang: Unicity Theorem and Strict Concvexity of or metric linear spaces, Tamakang J.Math. 11(1980).

49-51, Corrigendum, 14(1983), 103-104

[4]   T.D. Narang: A characterization or Strictly convex Metrix linear Space, 39 (1986), 149-151.

[11] Neera : On Strictly Convex Normed Linear Spaces, M.Phil. Dissertation ,G.N.D.University (1985)

[6] R.R.Phelps : Convex Sets and Nearest Points , Proc. Amer . Math. Sec. 8 (1975), 790 -797

[7] Walter Rudin : Functional Analysis , Tata Mc Graw – Hill (1973)

[8] K.P.R. Sastry and S.V.R. Naidu : Convexity Conditions in Metric Linear Spaces , Mathematics Seminar Notes 7 (1979), 235-251

[9] K.P.R. Sastry and S.V.R. Naidu : Uniform Convexity and Strict Convexity

in Metric Linear Spaces, Math . Nachr., 104(1981),331-343

[10] K.P.R. Sastry and S.V.R. Naidu : Proximity and Ball Convexity in Metric Linear Spaces, Tamkang J. Math 14 (1983) 5-7

***Dr. Meenu Sharma**
Principal
A.S. College for Women, Khanna

****