MODULE* – 6

ON NON-EXPANSIVE RETRACTS

In order to generalize a theorem of Belluce and kirk [1] on the existence of a common fixed point of a finite family of commuting non-expansive mappings, Ronald E. Bruck Jr.. [3] studied some properties of fixed-point sets of non-expansive mappings in Banach spaces. In this paper, we extend some results of [3] to convex metric spaces. We also prove that the fixed point set of a non-expansive mapping satisfying conditional fixed point property (CFP) is a non-expansive retract and hence metrically convex.

To start with, we recall a few definitions.

Definition 6.1 [3]. Let (X,d) be a metric space and C a closed convex subset of X. A mapping T:C$\rightarrow$X is said to satisfy conditional fixed point property (CFP) if either T has no fixed point or T has a fixed point in every non-empty bounded closed convex [3].

Definition 6.2 [3] A set C is said to have the **fixed point property (f.p.p.)** for non-expansive mappings if every non-expansive mapping of C into C has a fixed point.

Definition 6.3 A subset of a metric space (x,d) is said to be **matrically convex** [4] if for each pair of distinct points x_0 , x_1 of F, there exists a point y in F distinct from x_0 and x_1 such that d(x_0 x_1) =d(x_0 y)+d(y, x_1).

The following theorem shows that the non-expansive retracts are metrically convex.

Theorem 8.1 [5]. Let C be a non-empty convex subset of a convex matric space (X,d). Then non-expansive retracts of C are metrically convex.

Proof. Let r be a non-expansive retraction of C onto F and x_0 , x_1 be two distinct points of F. For t in [0,1], consider W(x_0 x_1 ,t) $\equiv x_t \in$. C Since r(x_0)=$x_0 \neq x_1$, by the continuity of W and r the mapping t $\rightarrow$ r(x_t) is distinct from x_0 and x_1 .

Let y be one such r (x_t). Then $y \in F$ is distinct from x_0 and x_1 while

$$
\begin{aligned}
d(x_0, x_1) &= d(r(x_0), r(x_1)) \\
&\le d(r(x_0), r(x_t)) + d(r(x_t), r(x_1)) \\
&\le d(x_0, x_t) + d(x_t, x_1) \text{ (r being non-expansive)} \\
&= d(x_0, w(x_0, x_1, t)) + d(w(x_0, x_1, t), x_1) \\
&\le t\, d(x_0, x_0) + (1-t)\, d(x_0, x_1) + t d(x_0, x_1) \\
&\quad +(1-t) d(x_1, x_1) \text{ (by the convexity of X)} \\
&= d(x_0, x_1).
\end{aligned}
$$

So, equality holds throughout and in particular,

$$d(r(x_0), r(x_1)) = d(r(x_0), r(x_t)) + d(r(x_t), r(x_1))$$

i.e. d (x_0 , x_1) =d(x_0 , y) +d(y, x_1) . Hence F is metrically convex.

Next we shall show that the fixed point set of a non-expansive mapping satisfying (CFP) is a non-expansive retract of C and hence metrically convex. To achieve this, we firstly prove a few lemmas:

Lemma 6.1. [5]. Suppose F is a non-empty subset of a locally compact set C in a metric space (X,d) and let N(F)={f : f:C$\rightarrow$C is non-expansive and fx=x for all

$x \in F$ }. Then N(F) is compact i.e. the set of all non-expansive retractions of C onto F is compact.

Proof. Fix $x_0 \in F$ and for $x \in C$ define

$$C_x = \{ y \in C : d(y, x_0) \leq d(x, x_0) \}.$$

For each x in C and f in N(F), $f(x) \in C_x$ since $f(C) \subset C$ and

$$d(fx, x_0) = d(fx, fx_0) \leq d(x, x_0).$$

Thus by the definition of cartesian product of indexed family, N(F) is a subset of the cartesian product $P = \prod_{x \in C} C_x$.

Since C is locally compact, each C_x is compact and so by Tychonoff's theorem P is compact.

Now we show that N(F) is closed in P. Let $< f_n >$ be a sequence of elements in N(F) with $< f_n > \to f_0$ i.e. $f_n : C \to C$ is non-expansive and $f_n x = x$ for all $x \in F$ and for each n.

Then. $f_n x = x \Rightarrow \lim_{n \to \infty} f_n x = x \Rightarrow f_0 x = x.$

Also,

$$d(f_0 x, f_0 y) = d(\lim_{n\to\infty} f_n x, \lim_{n\to\infty} f_n y)$$

$$= \lim_{n\to\infty} d(f_n x, f_n y)$$

$\leq d(x, y)$ (each f_n being non-expansive)

Therefore, N(F) is closed in P and hence compact.

Lemma 6.2. [5] , Suppose F is a non-empty subset of a locally compact set C in a metric space (X,d). Then there exists an r in N(F) = {f : f: C--->C is non-expansive and fx=x for all $x \in$ F} such that each f in N(F) acts as an isometery on the range of r i.e.

d(f(r(x)), f(r(y))) = d (r (x), r (y)) for all x , y $\in C$.

Proof. Define an order ≤ on N (F) by setting f≤g if

d(fx,fy) ≤ d(gx,gy) for all x,y in C with inequality holding for atleast one pair of x, y.

Clearly ≤ is a partial order on N(F).

For each f in N(F), we define M={ $g \in$ N(F) : g≤f }.

Clearly M is closed in N(F) (Let $< g_n >$ be a sequence of

elements in M with $< g_n > \to g_0$. So $g_n : C \to C$ is non-expansive, $g_n x = x$ for all $x \in F$ and $g_n \le f$. As discussed in Lemma 6.1 ; we have $g_0 : C \to C$ is non-expansive with $g_0 x = x$ and also $g_n \le f$ implies $\lim_{n \to \infty} g_n \le f$ and hence $g_n \in C$). Therefore M is closed in N(F) and hence compact as N(F) is compact.

It follows that (N(F), ≤) contains a minimal point. Indeed by zorn's lemma, it is sufficient to show that whenever $\{g_\lambda : \lambda \in \Lambda\}$ is linearly ordered by ≤ , then there exists g in N(F) with g ≤ g_λ for all λ. But if the g_λ s are linearly ordered by ≤, family $\{M(g_\lambda) : \lambda \in \Lambda\}$ is linearly ordered by inclusion. Since each $M(g_\lambda)$ is compact and non-empty, there exists $g \in \cap M(g_\lambda)$ i.e. there exists g in N(F) with $g \le g_\lambda$ for each λ.

Hence there exists a minimal element r of N(F) It si easily verified that for $f \circ r \in$N(F) whenever $f \in$ N (F)

(i) $d(f \circ r(x), f \circ r(y)) = d(f(r(x)), f(r(y)))$

$$\leq d(r(x), r(y))$$

and

(ii) $(f \circ r)(x) = f(r(x)) = f(x) = x$ or $f \circ r \in$N(F).

Now in particular $d(f \circ r(x), f \circ r(y)) \leq d(r(x), r(y))$ (6.1)

If inequality holds in (6.1) for any pair x , y in C then $f \circ r < r$ while for $f \circ r \in$N(F), contradicting the minimality of r in N(F). Therefore, equality holds in (6.1) for each x , y in C.

i.e. d(f (r (x)) , f (r (y))) = d (r (x), r (y)) for all x, y$\in C$.

Using Lemmas 6.1 and 6.2, non-expansive retracts of C can be described as:

Lemma 6.3 [5]. Suppose C is locally compact and F a non- empty subset of C in a metric space (X,d). Suppose that for each z in C there exists h in N(F) such that h(z) $\in$ F. Then F is a non-expansive retract of C.

Proof. Since F is a non-empty subset of a locally compact set C, by Lemma 6.2, there exists an r in N(F) = { f : f : C$\rightarrow$C is non-expansive and fx=x for all x $\in$ F } such that each element of N(F) acts as an isometery on the range of r. Since r$\in$ N(F) we get r:C$\rightarrow$C is non-expansive and rx=x for all x$\in$ F. So, to show that r is a non-expansive retraction of C onto F, it is sufficient to show that r(x) $\in$ F for each x in C.

Applying the given hypothesis to the point z=r(x), there exists h in N(F) with h (r(x)) $\in$ F. Let y =h(r(x)). By Lemma 6.2,

$$d(h(r(x)), h(r(y))) = d(r(x), r(y)) \qquad (6.2)$$

Since y$\in$F and h, r$\in$N (F) , we have h (y) = y and r (y) = y and hence y=h (r (y)) while y=h (r (x)) by definition. So it follows from (6.2) that r(x) = r (y) . Since r (y) = y, r(x) = y$\in$F for each x$\in$C. Therefore r : C$\rightarrow$F is a non-expansive retraction and hence F is a non-expansive retract of C.

It was show in [2] that if C is a closed convex subset of the real, reflexive, strictly convex Banach space X and T: C$\rightarrow$C is non-expansive then F(T) is a non-expansive retract of C. The same conclusion was drawn in [3] when C is a non-empty closed convex, locally weakly compact subset of a Banach space and T:C$\rightarrow$C is a non-expansive mapping satisfying (CFP). In convex metric spaces we have.

<u>Theorem 6.2</u> [5] Let C be a locally compact, convex set in a convex metric space (X,d) with properties (I) and (I^*) and T:C$\rightarrow$C a non-expansive mapping satisfying (CFP) then F(T) is a non-expansive retract of C.

<u>Proof</u>. Since the empty set, by definition, is a non-expansive retract of C, we assume that F(T) $\neq \phi$.

Fix z in C and define k = { f (z) : f$\in$N (F (T))).

Clearly K is the image of N(F(T)) under the z^{th} co-ordinate projection map of P onto C_z. Since N(F(T)) is compact (Lemma 6.1) and the projection continuous, K must be compact and hence bounded. Clearly K is non-empty. We claim that K is convex.

Let f, g $\in$N (F(T)) . and $0 \le \lambda \le 1$. Since N(F (T)) =

{ f : f : C$\to$C is non-expansive and fx=x for all x $\in$F (T)), We are to show that

(i) W (f,g,λ) (x) = x for all x$\in$F (T) and

(ii) W (f, g,λ) : C$\to$C is non-expansive

(i) Let x $\in$F (T) Consider

$$d(W(f, g, \lambda)(x), x) = d(W(f(x), g(x), \lambda), x)$$

$$\leq \lambda d(f(x), x) + (1-\lambda) d(g(x), x))$$

(by the convexity of X)

$$= \lambda d(x, x) = (1-\lambda)d(x,x) \text{ (as f, g} \in \text{N (F(T)))}$$

$$= 0$$

implying W(f , g ,λ) (x) = x for all x $\in$ F(T). Consider

$$d(W(f,g,\lambda)(x), W(f,g,\lambda)(y))$$

$$= d(W(f(x), g(x), \lambda), W(f(y), g(y), \lambda))$$

$$\leq d(w(f(x), g(x), \lambda), w(f(x), g(y), \lambda))$$

$$+d(w(f(y), g(y), \lambda), w(f(x), g(y), \lambda))$$

$$\leq (1-\lambda) d(g(x), g(y)) + \lambda d(f(y), f(x))$$

(by properties (I^*) and (I))

$$\leq (1-\lambda) d(x, y) + \lambda d(x, y)$$

(as f, g are non-expansive)

$$= d(x, y)$$

implying that W (f , g ,λ) is non-expansive. Therefore k is a bounded, non empty, closed and convex subset of C.

Since $T \circ f \in$ N (F (T)) whenever f$\in$N (F (T)) as

$$d(T \circ f(x), T \circ f(y)) = d(T(f(x)), T(f(y)))$$

$$= d(Tx, Ty) \quad (\text{as } f(x) = x \text{ and } f(y) = y)$$

$$\leq d(x, y) \quad (\text{as T is non-expansive})$$

and $\quad T \circ f(x) = T(f(x))$

$$= T(x) \quad (\text{as } f(x) = x)$$

$$= x \text{ for all } x \in F(T),$$

we have T (K) $\cap$ K.

Since T satisfies (CFP) and K is a non-empty bounded closed convex set that T leaves invariant, T has a fixed point in K. So there exists h in N (F(T)) with h (z) in F(T) .

Since this is so for each z in C, by Lemma 6.3, F(T) is a non expansive retract of C.

Corollary 6.1 [5]. Let C be a non-empty locally compact convex subset of a convex metric space (X,d) satisfying properties (I) and (I^*), T: C $\rightarrow$ C be non-expansive mapping satisfying (CFP) . Then F (T) is metrically convex.

Proof. By Theorem 6.2, F (T) is a non expansive retract of C and hence by Theorem 6.1 , it is metrically convex.

The following lemma will be used in proving our next result.

Lemma 6.4 [5]. Suppose C is a convex subset of a strictly convex metric space (x,d) with properties (I) and (I^*) and T_1, T_2,: C $\rightarrow$ C be non-expansive mappings such that $F(T_1) \cap F(T_2) \neq \phi$. Then exists a non-expansive mapping T:C $\rightarrow$ C such that F (T) = $F(T_1) \cap F(T_2)$.

Proof. Define T : C $\rightarrow$ C as

$$T(x) = W(T_1, T_2, \lambda)(x) = W(T_1(X), T_2(X), \lambda) \text{ for } 0 \le \lambda \le 1.$$

We claim that T is the required mapping.

Consider

$$d(Tx, Ty) = d(W(T_1, T_2\lambda)(x), W(T_1, T_2, \lambda)(y))$$

$$d(W(T_1(x), T_2(x), \lambda), W(T_1(y), T_2(y), \lambda))$$

$$= d(W(T_1(x), T_2(x), \lambda), W(T_1(x), T_2(y), \lambda))$$

$$+ d(W(T_1(x), T_2(y), \lambda), W(T_1(y), T_2(y), \lambda))$$

$$\le (1-\lambda) d(T_2(x), T_2(y)) + \lambda d(T_1(x), T_1(y))$$

(by properties (I^*) and (I))

$\le (1-\lambda)\, d(x, y) + \lambda\, d(x, y)$

(as T_1 , T_2 are non-expansive)

$= d(x,y)$

implying that T is a non-expansive mapping on C.

Now we show that $F(T) = F(T_1) \cap F(T_2)$. Let $x \in F(T_1) \cap F(T_2)$ i.e. $T_1(x) = x$ and $T_2(x) = x$. Then $T(x) = W(T_1(x), T_2(x), \lambda) = W(x, x, \lambda) = x$ as $d(W(x,x,\lambda), x) \le \lambda d(x, x) + (1-\lambda) d(x, x) = 0$. Therefore $x \in F(T)$ $F(T_1) \cap F(T_2) \subseteq F(T)$.

Now suppose $x \in F(T)$ and $y_o \in F(T_1) \cap F(T_2)$.

Consider

$$d(x, y_0) = d(Tx, y_0)$$

$= W(T_1(x), T_2(y), \lambda), y_0)$

$\leq \lambda \, d(T_1(x), y_0) + (1-\lambda) \, d(T_2(x), y_0)$

(by the convexity of X)

$= \lambda \, d(T_1(x), T_1(y_0)) + (1-\lambda) \, d(T_2(x), T_2(y_0))$

(as $y_o \in F(T_1) \cap F(T_2)$)

$\leq \lambda \, d(x, y_0) + (1-\lambda) \, d(x, y_0)$

(as T_1, T_2 are non-expansive)

$= d(x, y_0)$.

Thus equality holds throughout and so

$$d(W(T_1(x), T_2(x), \lambda), y_0) = d(x, y_0).$$

Since $d(T_1(x), y_0) \leq d(x, y_0)$ and $d(T_2(x), y_0) \leq d(x, y_0)$,

the strict convexity of X implies $T_2(x) = T_1(x)$. Therefore

$x = Tx$

$= W(T_1(x), T_2(x), \lambda)$

$= W(T_1(x), T_1(x), \lambda)$

$= T_1(x)$

Thus $x = T_1(x) = T_2(x)$ and so $x \in F(T) = F(T_1) \cap F(T_2)$ implying

F(T)$\subseteq$ $F(T_1) \cap F(T_2)$.

It was shown in [2] that for a closed convex subset C in a real, reflexive, strictly convex Banach space X , the class of non-expansive retracts of C is closed under arbitrary intersection. The same conclusion was drawn in [3] when C is a non empty closed, convex, locally weakly compact subset of a strictly convex Banach space X . In convex metric spaces we have:

<u>**Theorem 6.3**</u> [5]**.** Suppose (X , d) is a complete strictly convex metric space with properties (I) and (I^*), C a locally compact, convex set in X and T_1 , T_2: C $\rightarrow$ C be non-expansive mappings. Then $F(T_1) \cap F(T_2)$ is a non-expansive retract of C and hence metrically convex.

Proof. If F (T_1)$\cap$ F (T_2) $= \phi$, then clearly F (T_1)$\cap$ F (T_2) is a non-expansive retract of C and thence by Theorem 6.1 is metrically convex.

Now suppose F (T_1) $\cap$ F (T_2) $= \phi$ then by Lemma 6.4, there exists a non-expansive mapping T:C $\rightarrow$C defined by

T(x) = W (T_1, T_2, λ) (x), $0 \leq \lambda \leq 1$ such that f(T) = F (T_1)$\cap$ F (T_2). We show that F (T_1)$\cap$ F (T_2) = F (T) is a non-expansive retract of C.

Let x $\in$F (T) = F (T_1)$\cap$ F (T_2). Then T_1 (x) = x and T_2 (x) = x.

Consider

d(Tx, x) = d(w(T_1 (x) , T_2 (x) , λ) ,x)

$\leq \lambda$ d (T_1 (x) , x) + (1-λ) d(T_2 (x) ,x))

(by the convexity of X)

= λ d (x, x) + (1-λ) d (x, x)

= 0

and so Tx = x i.e. x is a fixed point of T in C.

Let K be a bounded closed convex set in C such that T leaves K invariant.

Now $T^* = T/K : K \to K$ is non-expansive as T:C$\to$C is a non-expansive mapping.

Since every convex set is starshaped with respect to each of its elements so is K . Let p be a starcentre of K and $T_p : K \to K$ be a mapping defined by $T_p(x) = p$ for all $x \in K$.

Let $< k_n >$ be any sequence of real numbers with $0 \le k_n < 1$ and $k_n \to 1$.

Define $T' : K \to K$ by $T'(x) = W(T', T_p, k_n)(x)$ for all $x \in K$.

Consider

$$d(T'x, T'y) = \mathrm{d}(\mathrm{W}(T^*, T_p, k_n)(\mathrm{x}), \mathrm{W}(T^*, T_p, k_n)(\mathrm{x})$$

$$= d(W(T^*(x), p, k_n), W(T^*(y), p, k_n))$$

$$\le k_n d(T^*(x), T^*(y)) \qquad \text{(by property (I))}$$

$$\le k_n d(x, y) \qquad (T^* \text{ being non-expansive})$$

Thus T' is a k_n -contraction on K and so by Banach contraction principle, T' has a fixed point, say x_1 in K i.e $T' x_1 = x_1$.

Now,

$$\mathrm{d}(x_1, T' x_1) = \mathrm{d}(T' x_1, T' x_1)$$

$$= d(W(T', T_p, k_n)(x_1), T'(x_1))$$

$$= d(W(T'(x_1), p, k_n), T'(x_1))$$

$$\le k_n d(T^*(x_1), T^*(x_1)) + (1 - k_n) d(p, T^*(x_1)) \qquad \text{(by the convexity of X)}$$

$$\to 0 \text{ as } n \to \infty.$$

So for $x_1 \in K$, we have $T^*(x_1) = x_1$ i.e. $T^* = T/K$ has a fixed point in K implying that T has a fixed point x_1 in every non-empty closed convex set K that T leaves invariant i.e. T satisfies (CFP). Thus we get a non-expansive mapping T:C $\to$ C satisfying (CFP). So, by Theorem 6.2, F(T) is a non-expansive retract of C and hence by Theorem 5.1, $F(T_1) \cap F(T_2) = F(T)$ is metrically convex.

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