MODULE* – 5

FIXED POINTS AND APPROXIMATION

This module deals with the structure of fixed point set, the problem of invariant approximation and an application of fixed point theorem to $\varepsilon - s$imultaneious approximation.

It is known (see e.g. [4] Theorem 6, p. 243) that for a closed convex subset K of a strictly convex normed linear space X and a non-expansive mapping T:K--->X, the fixed point set (possibly empty) of T is a closed convex set. We extend this result to pseudo strictly convex metric linear spaces in the first section.

Fixed points of non-expansive mappings have been extensively discussed in strictly convex normed linear spaces (see e.g. [8]. Using fixed point theory, Meinardus [10] and Brosowski [5] established some interesting results on invariant approximation in normed linear spaces. Later various researchers obtained generalizations of their results (see e.g. [8] and the references cited therein). In the second section we extend and generalize the work of Brosowski [5], Hicks and Humphries [7], Khan and Khan [9] and Singh [15], [16] to metric spaces having convex strucrure and to metric linear spaces having strictly monotone metric (a notion introduced by Guseman and Peters [6]). We have proved the existence of an invariant point x_o in the set $P_C(x)$ satisfying certain conditions. We have also established a result on invariant approximation in strictly convex metric spaces in this section.

Some applications of fixed point theorems to best simultaneous approximation were given by Ismat Beg and Naseer Shahzad [7], R.N. Mukherjee and V. Verma [11] and few others when the underlying spaces are normed linear spaces. Using a result of Beg and Azam [1] on fixed points of mutivalued mappings, we give an application of a fixed point theorem to $\varepsilon -$simultaneous approximation when the spaces are convex metric spaces in the third section of this chapter.

5.1 Fixed Points in Pseudo Strictly Convex Spaces

The following theorem give the structure of the fixed point set of a non-expansive mapping in pseudo strictly convex metric spaces:

Theorem 5.1.1 [13] : Let K be a closed convex subset of a convex metric linear space (X, d) with pseudo strict convexity and T : K $\to$ X, a non-expansive mapping. Then fixed point set (possibly empty) of T is a closed, convex set.

Proof: Let $F = \{x \in K : Tx = x\}$ be the fixed point set of T. Firstly we prove the closedness of the set F. Let x be a limit point of F. then there exists a sequence $\{x_n\}$ in F such that $\{x_n\} \to x$. Since a non-expansive mappint is always continuous, we get $Tx_n \to Tx$. Also $Tx_n = x_n \to x$. as $x_n \in F$ and so Tx=x i.e. $x \in F$. Hence F is a closed set.

Now we show that F is convex. Let x, $y \in F$ and $\lambda \in [0, 1]$ Then x, $y \in K$ and so $\lambda x + (1-\lambda) y = z$ (say) $\in K$ Consider

$$d(x, Tz) = d(Tx, Tz)$$

$\leq$ d (x, z) (as T is non-expansive)

$=$ d(x, λx+(1-λ) y)

$\leq$ λ d (x, x) + (1- λ) d (d, y)

(by the convexity of X)

$=$ (1-λ) d (x, y).

Also, d (Tz, y) = d (Tz, Ty)

$\leq$ d (z, y) (as T is non-expansive)

$=$ d (λ x + (1-λ) y, y)

$\leq$ λ d (x, y) + (1-λ) y, y)

(by the convexity of X)

$=$ λ d (x, y).

Therefore, d (x, Tz) + d (Tz, y) $\leq$ d (x, y).

Also by the triangle inequality,

d (x, y) $\leq$ d (x, Tz) + d (Tz, y).

Therefore

d (x, y) = d (x, Tz) + d (Tz, y)

i.e. d ((x-Tz) + (Tz-y), 0)=d (x-Tz, 0) +d (Tz-y, 0).

So by the pseudo strict convexity of X, we have

x-Tz=k (Tz-y) i.e. Tz=x/(1+k)+ky/(1+k) i.e. Tz$\in$[x, y]

Next we show that the non-expansivity of T implies Tz=z. Since z $\in$ [x, y] and Tz$\in$ [z, y], z will be either between x and Tz or between x and Tz or between Tz and y.

Suppose z lies between x and Tz then

$$d(Tz, x) = d(Tz, Tx) \text{ as } x \in F$$
$$\leq d(z, x) \text{ as T is non-expansive.}$$

$$\text{Now } d(Tz, z) + d(z, x) = d(Tz, x) \quad (\text{as } z \in [x, Tz])$$
$$\leq d(z, x)$$

implies d (Tz, z) = 0 and so Tz = z

Similarly, if z is between Tz and y, we shall get Tz=z. Therefore, Tz = z i.e. z$\in$ F i.e. $\lambda x + (1-\lambda) y \in F$, for all x, y$\in$F and $0 \leq \lambda \leq 1$. Hence F is a convex set.

5.2 Invariant Approximation

In this section, we extend and generalize some of the results of Brosowski [5], Hicks and Humphries [7], Khan and Khan [9] and Singh [15], [16] on invariant approximation in strictly convex metric spaces, in metric linear spaces having strictly monotone metric and in $\varepsilon-$ chainable convex metric spaces.

To start with we recall a few definitions.

Defintion 5.2.1 [6]. Let (X, d) be a metric linear space. The metric d for X is said to be strictly monotone [27] if $x \neq 0,\ 0 \leq t < 1$ imply d (tx, 0) <d (x, 0).

Definition 5.2.2. A metric linear space (X, d) is said to satisfy property (*) if

$$d(\lambda x+(1-\lambda)\, y, z) \le \lambda d(x, z) + (1-\lambda)\, d(y, z)$$

for every x, y, z $\in$ X and $0 \le \lambda \le 1$.

Clearly, every normed linear space satisfies property (*).

The following lemma in metric linear spaces satisfying property (*) will be used in the proof of Theorem 5.2.1.

Lemma 5.2.1 [14] Let (X, d) be a metric linear space satisfying property (*) , C a subset of X and $x \in X$. Then $P_C(x) \subset \partial C \cap C$ where ∂C is the boundary of C.

Proof. Let $y \in P_C(x)$. For each positive integer n, let $\lambda_n = n/(n+1)$.

Since $d(y, \lambda_n y+(1-\lambda_n)x) \le (1-\lambda_n)\, d(x, y)$ for all n (using property (*)), $\lim_{n\to\infty} [\lambda_n y + (1-\lambda_n)\, x] = y$. So each neighbourhood of y contains atleast one $\lambda_n y + (1-\lambda_n)\, x$. Also,

$d(y, \lambda_n y+(1-\lambda_n)x) \le \lambda_n d(y,x) < d(y, x)$ for all n implies that $\lambda_n y+(1-\lambda_n)\, x \notin C$ for any n i.e. y is not an interior point of C and so $y \in \partial C$. Also $y \in P_C(x)$ implies $y \in C$.. Thus, $y \in \partial C \cap C$ and hence $P_C(x) \subset \partial C \cap C$.

We have the following result on invariant approximation in metric linear spaces:

Theorem 5.2.1 [14]. Let (X, d) be a metric linear space with strictly monotone metric d and C a subset of X. Let T be a non-expansive mapping on $P_C(x) \cup \{x\}$ where x is a T-invariant point. Then there is a x_o in P_c (x) which is also T-invariant provided.

(a) $T: \partial C$ ---> C

(b) P_c (x) is nonempty, starshaped and compact.

(c) Either C is closed or (X, d) satisfies property (*)

Proof. Let P be starcentre of $P_C(x)$, then $\lambda x + (1-\lambda)p \in P_C(x)$ for every $x \in p_c(x),\ \ 0 \le \lambda \le 1$. We claim that $T: P_C(x) \to P_C(x)$.

Suppose (X, d) satisfies property (*) then by Lemma 5.2.1,

$P_C(x) \subset \partial C \cap C$. So for $y \in p_C(x)$, we get Ty $\in$ C as $T: \partial C$ --->C.

Suppose C is closed then $y \in P_C(x)$ implies $y \in \partial C$ leading to Ty $\in$ C as $T: \partial C$ --->C.

Thus in both the cases, Ty $\in$ C. Consider

d(x, Ty) = (Tx, Ty) (as x is a T-invariant point)

$\le$ d (x, y) (as T is non-expansive on $P_C(x) \cup \{x\}$)

= d (x, C)

$\le$ d (x, Ty).

This give d (x, Ty) = d (x, C) i.e. Ty $\in P_C(x)$ for $y \in P_C(x)$ and hence T: $P_C(x)$ ----> $P_C(x)$.

Let k_n, $0 \le k_n < 1$ be a sequence of real numbers such that $k_n \to 1$ as $n \to \infty$. Define T_n: $P_C(x)) \to P_C(x)$ as $T_n y = k_n\ Ty + (1-k_n)\ p$ for every $y \in p_C(x)$. Since T maps $p_C(x)$ into $P_C(x)$ for each n ($y \in p_C(x)$ and T: $P_C(x) \to P_C(x)$ imply $Ty \in p_C(x)$ and $P_C(x)$ being starshaped w.r.t. p we get $k_n Ty + (1-k_n)\ p \in P_C(x)$).

Also,

$$
\begin{aligned}
d(T_n x, T_n y) &= d(k_n Tx+(1-k_n)p, k_n Ty+(1-k_n)p) \\
&= d(k_n Tx, k_n Ty) \\
&= d(k_n(Tx-Ty), 0) \\
&< d(Tx-Ty, 0) \text{ (as d is strictly monotone)} \\
&= d(Tx, Ty) \\
&\le d(x, y) \text{ (as T is non-expansive on } P_c(x) \cup \{x\})
\end{aligned}
$$

Hence T_n is non-expansive on $P_C(x) \cup \{x\}$ for each n. Since $P_C(x)$ is compact and starshaped, T_n has a unique fixed point, say, x_n for each n ([6], Theorem 2) i.e. $T_n x_n = x_n$ for each n.

Since $P_C(x)$ is compact, $<x_n>$ has a convergent subsequence $x_{n_i} \to x_0 \in P_C(x)$. We claim that $Tx_0 = x_0$.

Consider $x_{n_i} = T_{n_i} x_{n_i} = k_{n_i} + (1-k_{n_i})\ p$. Taking limit as $i \to \infty$, we get $x_0 = x_0$ ($x_{n_i} \to x_0$ implies $T_{n_i} \to Tx_0$ as T is continuous on $P_C(x) \cup \{x\}$ i.e. $x_0 \in P_C(x)$ is T-invariant.

Since every normed liner space is a metric linear space with property (*) and the metirc induced by the norm is strictly monotone, we have:

Corollary 5.2.1 [5] . Let T be a non-expansive linear operator on a normed linear space X. Let C be a T-invariant subset of X and x a T-invariant point. If the set of best C-approximants to x is non-empty, compact and convex, then it contains a T-invariant point.

Corollary 5.2.2 [15]. Let T be a non-expansive mapping on a normed linear space X. Let C be a T-invariant subset of X and x_o a T-invariant point in X. If D, the set of best C-approximants to x_o is non-empty, compact and starshaped, then it contains a T-invariant point.

Corollary 5.2.3 [16] . Let X be a normed linear space and T:X--->X a mapping. Let C be a subset of X such that C is T-invariant and let x_o be a T-invariant point in X. If D, the set of best C-approximants to x_o is non-empty, compact and starshaped and T is

(i) continuous on D

(ii) $\|x\text{-}y\| \leq d(x_0, C) \Rightarrow \|Tx - Ty\|$ for x, y in $D \cup \{x\}$,

then it contains a T-invariant point which is a best approximation to x_o in C.

Note: The continuity of T on D need not be assumed as it follows from (ii)

Since each p-norm generates a translation invariant metric d satisfying property (*) and is strictly monotine, we have:

Corollary 5.2.4 [9]. Let $(E, \|\cdot\|_p)$ be a p-normed space, T:D-->E a non-expansive mapping with a fixed point $u \in E$ and C a closed T-invariant subset of E such that T is compact on C. If $P_C(u)$ is starshaped, then there exists an element in $P_C(u)$ which is also a fixed point of T.

In strictly convex metric spaces, we have the following result on invariant approximation:

Theorem 5.2.2 [14]. Let (X, d) be a strictly convex metric space and T a non-expansive mapping on $P_C(x) \cup \{x\}$ where x is a T-invariant point. Let C be a subset of X, $T: \partial C \text{--->} C$ and $P_C(x)$ be non-empty and starshaped with starcentre q. Then $P_C(x) = \{q\}$ with Tq=q.

Proof. Let $p \neq q \in P_C(x)$. Then d (x, p) = d (x, q) = d (x, C). Since $p \neq q$, strict convexity of the space implies d (x, W (p, q, λ))<
dist (x,C) and so $W(p, q, \lambda) \notin P_C(x)$, $0 \le \lambda \le 1$. Starshapedness of $P_C(x)$ therefore implies p=q i.e. $P_C(x) = \{q\}$. Since X is convex, $P_C(x) \subset \partial C \cap C$ (Lemma 3.2 [17]). So for $y \in P_C(x)$, we get $Ty \in C$ as $T: \partial C \text{--} C$. Consider

$$
\begin{aligned}
d(x, Ty) &= d(Tx, Ty) \quad \text{(as x is a T-invariant point)} \\
&\le d(x, y) \quad \text{(as T is non-expansive on } P_C(x) \cup \{x\}) \\
&= d(x, C) \\
&\le d(x, Ty).
\end{aligned}
$$

This gives d (x, Ty) = d (x, C) i.e. $Ty \in P_C(x)$ for $y \in P_C(x)$ and so $T: P_C(x) \text{---->} P_C(x)$. Hence $Tq \in P_C(x) = \{q\}$, i.e. Tq=q.

5.3 Fixed Points and ε- **Simultaneous** Approximation

Using a result of Beg and Azam [1] on fixed point of multivalued mappings, we give an application of a fixed point theorem to ε- Simultaneous approximation in convex metric spaces in this section. We start with a few definitions.

Defintion 5.3.1. A metric space (X, d) is said to be ε-chainable (see e.g. [3]) if given x, $y \in X$, there is an ε-chain from x to y (i.e. a finite set of points $x=z_0, z_1, z_2,\ldots,z_n=y$ such that $d(z_{i=1}, z_i) < \varepsilon$ for all i= 1,2, ..., n.

Definition 5.3.2. A mapping T:X --> CB(X) is called an (ε, λ)- uniformly locally contractive mapping (where ε>0 and $0<\lambda<1$) (see e.g.[3] if x, y∈X and d(x, y) < ε then H (Tx, Ty) $\leq \lambda d$ (x, y) where H stands for Hausdorff metric on CB(x).

An application of a fixed point theorem to b.s.a. was given by Beg and Shahzad [2]. Using the following simplified version of a result due to Beg and Azam [1]:

Lemma 5.3.1 Let (x,d) be a complete ε-chainable metric space T:X -->CB(X) satisfies the condition:

$0<d(x, y) <\varepsilon$ implies $H(Tx, Ty) < kd(x, y)$

where $k \in [0,1[$, then there exists a fixed point of T.

We now extend the result of [2] to ε-s.a.

Theorem 5.3.1[12]. Let (x, d) be an ε-chainable convex metric space satisfying condition (I) and T:X -->CB (X) be a multivalued mapping. Let $G \in CB(X)$. For $F \in CB(X)$, if $cent_G(F,\varepsilon)$ is compact, starshaped, T-invariant and T is

(i) continuous on $cent_G(F,\varepsilon)$ and

(ii) $d(x, y) \leq H(F, G)$ implies $H(Tx, Ty) \leq d(x, y)$ for all x, y in $cent_G(F,\varepsilon) \cup F$, then $cent_G(F,\varepsilon)$ contains a T-invariant point.

Proof. The proof of this theorem is a minor modification of the one given by beg and Shahzad [3] for b.s.a. in normed linear spaces.

Let p be the starcentre of $cent_G(F,\varepsilon)$ then $W(x, p,\lambda) \in cent_G(F,\varepsilon)$ for each $x \in cent_G(F,\varepsilon)$. Let $<k_n>$ be a sequence of real numbers with $0 \le k_n < 1$ converging to 1.

Define T_n: $cent_G(F,\varepsilon)$ --->$CBcent_G(F,\varepsilon)$) as

$$T_n x = W(Tx, p, k_n) = \bigcup_{y \in T} W(y, p, k_n) \text{ for all } x \text{ in } cent_G(F,\varepsilon).$$

Clearly T_n is well defined as for $g_0 \in cent_G(F,\varepsilon)$ we have

$d_F(W(Tg_0, p, kn)) = d_F(W(g_0, p, kn))$ as $g_0 \in cent_G(F,\varepsilon)$ is T-invariant

$\le D(F,G)+\varepsilon$ as $cent_G(F,\varepsilon)$ being starshaped,

$$W(g_0, p, kn) \in cent_G(F,\varepsilon)$$

i.e. for $g_0 \in cent_G(F,\varepsilon)$, $T_n g_0 = W(Tg_0, p, k_n) \in CB(cent_G(F,\varepsilon))$.

Using condition (I), we have

$$H(T_n x, T_n y) = H(W(Tx, p, k_n), W(Ty, p, k_n))$$

$$\le k_n H(Tx, Ty)$$

$$\le k_n d(x, y)$$

for all $d(x, y) \le H(F, G)$ which implies that T_n is a $(H(F, G), k_n)$ uniformly local contraction for each n=1, 2, 3, ... it follows by Lemma 5.3.1 that each T_n has a fixed point, say, x_n. Since $cent_G(F,\varepsilon)$ is given to be compact, $\{x_n\}$ has a convergent subsequence x_{ni}--->z and so continutiy of T on $cent_G(F,\varepsilon)$ implies Tx_{ni} ---> Tz.

$$x_{ni} \in T_{ni} x_{ni} = W(T_{ni}, P, k_{ni})$$

Since k_{n2}--->1, $z \in Tz$ (as k_{ni} --->1 implies $z \in W(Tz, p, 1) = Tz$).

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