MODULE* – 4

ON ε-SIMULTANEOUS APPROXIMATION AND BEST SIMULTANEOUS CO-APPROXIMATION

This module deals with ε-Simultaneous Approximation and Best Simultaneous Co-Approximation. The chapter has been divided into two sections. In the first section we discuss ε-simultaneous approximation. The problem of best simultaneous opprozined (b.s.a.) is concerned with approximating simultaneously any two elements x_1, x_2 of a space X by the elements of a subset A of X. More generally, if a set of elements B is given in X, one might like to approximate all the elements of B simultaneously by a single element of A. This type of problem arises when a function being approximated is not known precisely, but is known to belong to a set. C. B. Dunhan [3] seems to have been the first who studied the problem of b.s.a. in normed linear spaces. R.C. Buck [2] studied the problem of ε-approximation which reduces to the problem of best approximation for the particular case when ε=0. In this section, we discuss ε-simultaneous approximation for any two elements x_1, x_2 and for a non-empty bounded subset F of a convex metric space (X, d) with respect to a non-empty subset G of X. Defining ε-simultaneous approximation map

$P_{G(\varepsilon)}: X \times X \to 2^G$ by $P_{G(\varepsilon)}(x_1, x_2) = \{g_0 \in G : d(x_1, g_0) + d(x_2, g_0) \leq r + \varepsilon\}$ where r=inf (d (x_1, g) + d (x_2, g) : g$\in$G} and we prove the upper semi-continuity of the map $P_{G(\varepsilon)}$. We also prove the convexity, boundedness, closeness and sharshapedness of the set $P_{G(\varepsilon)}(x_1, x_2)$ and of $P_{G(\varepsilon)}$(F) in the first section.

The second section of this module deals with Best Simultaneous Co-approximation A new kind of approximation, called best co-approximation was introduced in normed linear spaces by C. Franchetti and M. Furi [4] in 1972. This study was taken up later by T.D. Narang, P.L. Papini, Geetha S. Rao, Ivan Singer, S.P. Singh and few others (see [7], [8]. Generalizing the concept of best approximation, Geetha S. Rao and R. Sarvanan studied the problem of best simultaneous co-approximation in normed liner spaces in [10]. In this section, we study the problem of best simultaneous co-approximation in convex metric linear spaces and convex metric spaces thereby extending some of the results proved in [10] We have also given some properties of the set $S_G(x, y)$ i.e. the set of all best simultaneous co-approximations to x, y in G. We have proved that for a convex metric space (X, d) and G a convex subset of X, the set $S_G(x, y)$ is convex. We have also proved the upper semi continuity of the mapping $S_G : \{(x, y) : x, y \in X\} \to 2^G$ in totally complete metric linear spaces (a notion introduced by T.D. Narang [9]).

4.1 ε-Simultaneous Approximation

This section deals with ε-simultaneous approximation in metric spaces. In this section, we discuss ε-simultaneous approximation of any two elements x_1, x_2 and then of a non-empty bounded subset F of a convex metric space (X, d) with respect to a non-empty subset G of X.

To start with, we recall a few definitions.

Definition 4.1.1. Let (X, d) be a metric space and G a non-empty subset of X. An element $g_0 \in G$ is said to-be (i) an element of ε-approximation (ε-a.) to $x \in X$ if

$$d(x, g_0) \le d(x, g) + \varepsilon \text{ for all } g \in G \text{ and } \varepsilon > 0$$

$$\text{i.e. } d(x, g_0) \le \inf\{d(x, g) : g \in G\} + \varepsilon$$

The set of all ε-approximations to $x \in X$ from G is denoted by $P_{G(\varepsilon)}(x)$.

(ii) an element of ε-simultaneous approximation (ε-s. a.) to $x_1, x_2 \in X$ from G if $d(x_1, g_0) + d(x_2, g_0) \le r + \varepsilon$

where $r = \inf\{d(x_1, g) + d(x_2, g) : g \in G\}$

The set of all ε-smultaneous approximations to x_1 and x_2 from G will be denoted by $P_{G(\varepsilon)}(x_1, x_2)$.

Defintion 4.1.2. Let (X, d) be a metric space, G a non empty subset of X and F a non-empty bounded subset of X. For x in X, let

$$d_F(x) = \text{Sup}\{d(y, x) : y \in F\}$$

$$D(F, G) = \inf(d_F(x) : x \in G\}, \text{ and}$$

$$P_{G(\varepsilon)}(F) = \{g_0 \in G : d_F(g_0) \le D(F, G) + \varepsilon\} \text{ for } \varepsilon > 0$$

$$= \{g_0 \in G : \sup_{y\in F} \le \inf_{g\in G} \sup_{y\in F} d(y,g) + \varepsilon\}.$$

An element $g_0 \in P_{G(\varepsilon)}(F)$ is called ε-simultaneous approximation (ε-s.a.) of F with respect to G.

One of the advantages of considering the sets $P_{G(\varepsilon)}(x_1, x_2)$ and $P_{G(\varepsilon)}(F)$ with ε>0, instead of the sets $P_G(x_1, x_2)$ and $P_G(F)$ respectively is that the sets $P_{G(\varepsilon)}(x_1, x_2)$ and $P_{G(\varepsilon)}(F)$ are always non-void for ε>0.

The problem of ε-s.a: is equivalent to the problem of minimzing certain functional as shown below:

Lemma 4.1.1 [12]. If G is any subset of a metric space (X, d) and F a bounded subset of X. Then the functional ϕ:G-->R defined by ϕ (g) = $\sup_{f\in F}$ d (f , g) is continuous.

Proof. Let ε>0 be given. For any f$\in$F and g, g'$\in$G, we have d (f, g) $\le$d (f, g') +d (g' g) and so $\sup_{f\in F}$ d(f, g) $\le$ $\sup_{f\in F}$ sup d (f, g') + d (g', g)

i.e. ϕ: (g) -ϕ (g') $\le$d (g', g)

Inerchanging g and g', we get ϕ (g') -ϕ (g) $\le$d (g, g') and so $|\phi(g) - \phi(g')| \le d(g, g')$. Therefore, if d (g, g') <ε then $|\phi(g) - \phi(g')|$<ε and consequently ϕ is continuous.

If we take ϕ' (g) = ϕ (g) +ε then $\inf_{g\in G}$ ϕ' (g) $\inf_{g\in G}$ ϕ g) +ε.

So a $g_0 \in$G satisfying ϕ' (g_0)=$\inf_{g\in G}$ ϕ' (g) is an ε-s.a. to F

Thus, the problem of ε-s.a. is the problem of minimizing the functional ϕ' on G.

The following lemma deals with the boundedness and closedness of the set $P_{G(\varepsilon)}(F)$.

Lemma 4.1.2. [12] The set $P_{G(\varepsilon)}(F)$ is bounded and is a closed subset of G if G is closed. In addition, $P_{G(\varepsilon)}(F)$, is compact if G is compact.

Proof. Let g_0, $g_0{}' \in P_{G(\varepsilon)}(F)$. Then

$$d\,(g_0, g_0{}') \leq d\,(g_0, y) + d\,(y, g_0{}') \text{ for every } y \in F$$

$$\leq d_F\,(g_0) + d_F\,(g_0{}')$$

$$\leq D\,(F, G) + \varepsilon + D\,(F, G) + \varepsilon + D\,(F, D) + \varepsilon \text{ as } g_0,\ g_0{}' \in P_{G(\varepsilon)}(F)$$

$$= 2\,(D\,(F, G) + \varepsilon).$$

and so $P_{G(\varepsilon)}(F)$ is bounded.

Suppose G is closed. Let g_0 be a limit point of $P_{G(\varepsilon)}(F)$. Then there exists a sequence $<g_0^{(n)}>$ in $P_{G(\varepsilon)}(F)$ such that $<g_0^{(n)}> \to g_0$. Now $g_0^{(n)} \in P_{G(\varepsilon)}(F) \Rightarrow d_F\,(g_0^{(n)}) \leq D\,(F, G) + \varepsilon$ for all n

$\Rightarrow \lim d_F\,(g_0^{(n)}) \leq D\,(F, G) + \varepsilon) \Rightarrow d_F\,(g_0) \leq D\,(F, G) + \varepsilon \Rightarrow g_0 \in P_{G(\varepsilon)}(F)$ as G being closed. $g_0 \in G$. Therefore $P_{G(\varepsilon)}(F)$ is a closed subset of G. If the set G is compact then the set $P_{G(\varepsilon)}(F)$ is compact as closed subset of a compact set is compact.

If we take F = (x_1, x_2), we have:

Corollary 4.1.1 [12]. The set $P_{G(\varepsilon)}(x_1, x_2)$ is bounded and a closed subset of G if G is closed. In addition, $P_{G(\varepsilon)}(x_1, x_2)$ is compact if G is compact.

The following result shows the convexity of the set $P_{G(\varepsilon)}(F)$ in convex metric spaces.

Propostion 4.1.1. [14]. For any convex set G in a convex metric space (X, d), the set $P_{G(\varepsilon)}(F)$ is convex.

Proof. Let $g_0, g_0' \in P_{G(\varepsilon)}(F)$. Then $d_F(g_0) \leq$ D (F, G) + ε and $d_F(g_0') \leq$ D (F, G)+ ε. For any f∈F, consider d (f, W $(g_0, g_0', \lambda) \leq \lambda$ d (f, g_0) (1-λ) d (f, g_0'). This implies

$$
\begin{aligned}
d_F(W(g_0, g_0', \lambda)) &= \sup_{f\in F} d(f, W(g_0, g_0', \lambda)) \\
&\leq \lambda \sup_{f\in F} d(f, g_0) + (1-\lambda) \sup_{f\in F} d(f, g_0') \\
&= \lambda\, d_F(g_0) + (1-\lambda) \sup d(f, g_0') \\
&\leq \lambda\, (D(F,G) + \varepsilon) + (1-\lambda)(D(F,G) + \varepsilon) \\
&= D(F, G) + \varepsilon,
\end{aligned}
$$

Where W $(g_0, g_0', \lambda) \in$ G by the convexity of G, implying that W (g_0, g_0', λ) is ε-s.a. in G to F and so the set $P_{G(\varepsilon)}(F)$ is convex.

This proposition shows that if G is a convex subset of a convex metric space (X, d) and if g_0, g_0' are ε-s.a. in G to F then W (g_0, g_0', λ) is also ε-s.a. in G to F for every $\lambda \in$ I.

For F = $\{x_1, x_2\}$, we get:

Corollary 4.1.2 [15] For any convex set G in a convex metric space (X, d), the set $P_{G(\varepsilon)}(x_1, x_2)$ is convex.

The above corollary shows that in a convex metric space (X, d) if g_0, g_0' are ε-simultaneous approximations to x_1 and x_2 by elements of a convex set, then W (g_0, g_0', λ) is also ε-s.a. to x_1 and x_2 for every $\lambda \in$ I.

Next result proves the starshapedness of the set $P_{G(\varepsilon)}(F)$

Proposition 4.1.2 [11]. In a convex metric space (X, d), if G is starshaped with respect to g_0 and F a bounded subset of X then $P_{G(\varepsilon)}(F)$ is also starshaped with respect to g_0 provided $g_0 \in P_{G(\varepsilon)}(F)$.

Proof. Let $y \in P_{G(\varepsilon)}(F)$. Then $d_F(y) \le D(F, G) + \varepsilon$. Since G is starshaped with respect to g_0, $W(y, g_0, \lambda) \in G$ for $\lambda \in I$.

We claim that $W(y, g_0, \lambda) \in P_{G(\varepsilon)}(F)$ for all $\lambda \in I$.

Consider

$$\begin{aligned} d_F(W(y, g_0, \lambda) &= \sup_{f \in F} d(f, W(y, g_0, \lambda)) \\ &\le \sup_{f \in F} d(f, y) + (1-\lambda) \sup_{f \in F} d(f, g_0) \\ &= \lambda\, d_F(y) + (1-\lambda)\, d_F(g_0) \\ &\le \lambda\, (D(F,G) + \varepsilon) + (1-\lambda)(D(F,G) + \varepsilon) \\ &= D(F, G) + \varepsilon. \end{aligned}$$

Hence

$$d_F(w(y, g_0, \lambda) \le ((F,G) + \varepsilon$$

implying that $W(y, g_0, \lambda) \in P_{G(\varepsilon)}(F)$ for all $y \in P_{G(\varepsilon)}(F)$ and $\lambda \in I$ i.e. set $P_{G(\varepsilon)}(F)$ is starshaped with respect to g_0.

For F = (x_1, x_2), we get:

Corollary 4.1.3 [11]. In a conved metric space (X, d) if G is starshaped with respect to g_0 then $P_{G(\varepsilon)}(x_1, x_2)$, is also starshaped with respect to g_0 if $g_0 \in P_{G(\varepsilon)}(x_1, x_2)$.

Let CB (X) be the familyof non-empty closed and bounded subsets of X. Let H be a Hausdorff metric on CB (X) i.e. for

A, B $\in$ CB(X,

$$H(A,B) = \text{Max}\left(\sup_{a\in A} d(a,B), \quad \sup_{b\in B} d(b,A)\right)$$

Sastry and Naidu [12], and Govindrajulu [5] proved that under certain conditions, the best simultaneous approximation operator (not necessarily single-valued) is upper semi-continuous. The following result deals with the upper semi-continuity of the ε-simultaneous approximation map:

Proposition 4.1.3 [12]. If G is a compact subset of a metric space (X, d) then the ε-simultaneous approximation map $P_{G(\varepsilon)}$:X-->2^G is upper semi-continuous i.e. the set K={F $\in$ CB(X) : $P_{G(\varepsilon)}$ (F) $\cap N \neq \phi$} is closed for every closed set N in X.

Proof. Let {F_n} be a sequence in K converging to F $\in$ CB (X).

Then there exists a sequence <+x_n> in G such that $x_n \in P_{G(\varepsilon)}(F_n) \cap N$ for each n and so $d_F(x_n) \leq D(F_n, G) + \varepsilon$. Consider

$$d_F(x_n) \leq H(F, x_n) + d_F(x_n)$$

$$\leq H(F, F_n) + D(F_n, G) + \varepsilon \qquad (4.1.1)$$

Since G is compact, there exists a subsequence $< x_{n_i} >$ of <x_n> such that $< x_{n_i} >$ --> x_0 $x_0$and so (4.1.1) implies

$$d_F(x_0) \leq D(F,G) + \varepsilon \text{ as } H(F, F_n) \text{-->} 0$$

i.e. $x_0 \in P_{G(\varepsilon)}(F)$.

Since $< x_{n_i} > \in$ N and N is closed, $X_0 \in$ N. Consequently

$X_0 \in P_{G(\varepsilon)}(F) \cap$ N i.e. F$\in$K implying that the map $P_{G(\varepsilon)}$ is upper semi-continuous.

The following result deals with the upper semi-continuity of the ε-simultaneous approximation map $P_{G(\varepsilon)}(x_1, x_2)$:

Proposition 4.1.4 [11]. If G is a compact subset of metric space (X, d), then the ε-simultaneous approximation map

$P_{G(\varepsilon)}$: XxX-->2^G is upper semi-continuous i.e. the set B={ $(x_1, x_2) \in$ XxX: $P_{G(\varepsilon)}(x_1, x_2) \cap$ N $\neq \phi$} is closed for every closed set N$\subset$G.

Proof. The proof of Proposition 4.1.4 follows from Proposition 4.1.3 by taking F=(x_1, x_2). However, an independent proof is as under:

Let ($x_1{}^{(0)}$, $x_2{}^{(0)}$) be a limit point of the set B. Then there exists a sequence ($x_1{}^{(n)}$, $x_2{}^{(n)}$)> in B converging to ($x_1{}^{(0)}$, $x_2{}^{(0)}$) $\in$B, there exists a sequence <g_n> in G such that

$$g_n \in P_{G(\varepsilon)}(x_1^{(n)}, x_2^{(n)}) \cap \mathrm{N}, \quad n=1, 2,3 \ldots$$

Consider

$$\begin{aligned} d(x_1^{(0)}, g_n)+d(x_2^{(0)}, g_n) &\leq d(x_1^{(0)}, x_1^{(n)}) + d(x_1^{(n)}, g_n) + d(x_2^{(0)}, x_2^{(n)}) + d(x_2^{(n)}, g_n) \\ &\leq d(x_1^{(0)}, x_1^{(n)}) + d(x_2^{(0)}, x_2^{(n)}) + \inf\{d(x_1^{(n)}, g) + d(x_2^{(n)}, g) : g \in G\} + \varepsilon. \end{aligned}$$

$\leq \quad d\,(x_1^{(0)}, x_1^{(n)}) + d\,(x_2^{(0)}, x_2^{(n)}) +$

$\inf\{d\,(x_1^{(0)}, x_1^{(n)}) + d\,(x_1^{(0)}, g) + d(x_2^{(n)}, x_2^{(0)}) +$

$d\,(x_2^{(0)}, g) : g \in G\} + \varepsilon$

$= 2\,\{d\,(x_1^{(0)}, g) + d\,(x_2^{(0)}, g) : g \in G\} + \varepsilon\,.$

This implies

$\lim\,[d(x_1^{(0)}, g_n) + d\,(x_2^{(0)}, g_n)] \leq \inf\{d(x_1^{(0)}, g) : g \in G\} + \varepsilon$ (4.1.2)

Since G is compact, there exists a subsequence $< g_{n_i} >$ of $<g_n)$ such that $< g_{n_i} >$--> g_0 and so (4.1.2) implies

$d(x_1^{(0)}, + g_0) + d\,(x_2^{(0)}), g_0) \leq \inf\{d(x_1^{(0)}, g) + d\,(x_2^{(0)}, g) : g \in G\} + \varepsilon$

i.e. $g_0 \in P_{G(\varepsilon)}(x_1^{(0)}, x_2^{(0)})$.

Since $<g_n> \in N$ and N is closed, $g_0 \in N$. Consequently,

$g_0 \in P_{G(\varepsilon)}(x_1^{(0)}, x_2^{(0)}) \cap N$ i.e. $(x_1^{(0)}, x_2^{(0)}) \in B$. Thus B is closed and so $P_{G(\varepsilon)}$ is upper semi-continuous.

The next result deals with the structure of the sets

$P_{G(\varepsilon)}(x_1^{(0)}, x_2^{(0)})$.

Proposition 4.1.5 [11]. Let G be a non-empty compact subset of metric space (X, d) and $P_{G(\varepsilon)}$: XxX-->2^G($\equiv$ the collection of all bounded subsets of G) be the ε-simultaneous approximation map of XxX into G defined by $P_{G(\varepsilon)}(x_1, x_2) = \{ g_0 \in G : d(x_1, g_0)\,(x_0, g_0) \leq + \varepsilon\}$.

Where $r = \inf\{d(x_1, g) + d\,(x_2, g) : g \in G\}$.

Then the set $P_{G(\varepsilon)}$ (AxA) $=\cup\{P_{G(\varepsilon)}(x_1, x_2): x_1, x_2 \in A\}$ is compact for any copact subset A of X.

Proof. Let $<g_{n_i}>$ be any sequence in $P_{G(\varepsilon)}$ (AxA)$\subseteq$G. Since G is compact, there exists a subsequence $<g_{n_i}>$ of $<g_n>$ such that $<g_{n_i}> \text{--}> g_o \in G$

Since $<g_{n_i}>$ is a sequence in $P_{G(\varepsilon)}$ (AxA), there exist

$<x_1^{(n_i)}, x_2^{(n_i)}$ in AxA such that for each n_i,

$$g_{n_i} \in P_{G(\varepsilon)}(x_1^{(n_i)}, x_2^{(n_i)})$$

i.e. $d(x_1^{(n_i)}, g_{n_i}) + d(x_2^{(n_i)}, g_{n_i}) \leq \inf\{d(x_1^{(n_i)}, g) + d(x_2^{(n_i)}, g): g \in G\} + \varepsilon$ (*)

Since AxA is compact $<x_1^{(n_i)}, x_2^{(n_i)})>$ has a subsequence

$<x_1^{(n_{i_j})}, x_2^{(n_{i_j})}> \text{—}> (x_1^{(0)}, x_2^{(0)}) \in AxA$. Since

$\inf\{d(x_1^{(n_{i_j})}, g) + d(x_2^{(n_{i_j})}, g) : g \in G\} \text{--}> \inf\{d(x_1^{(0)}, g) + d(x_2^{(0)}, g): g \in G\}$

and

$$\begin{aligned} d(x_1^{(0)}, g_{n_{i_j}}) + d(x_2^{(0)}, g_{n_{i_j}}) &\leq d(x_1^{(0)}, x_1^{(n_{i_j})}) + d(x_1^{(n_{i_j})}, g_{n_{i_j}}) + \\ &\quad d(x_2^{(0)}, x_2^{(n_{i_j})}) + d(x_2^{(n_{i_j})}, g_{n_{i_j}}) \\ &\leq d(x_1^{(0)}, x_1^{(n_{i_j})}) + d(x_1^{(n_{i_j})}, g_{n_{i_j}}) + \\ &\quad \inf\{d(x_1^{(n_{i_j})}, g) + d(x_2^{(n_{i_j})}, g) : g \in G\} + \varepsilon \text{ (by (*))}. \end{aligned}$$

we have

$\lim [d((x_1^{(n_{i_j})}, g_{n_{i_j}}) + d(x_1^{(n_{i_j})}, g_{n_{i_j}})] \leq \inf(d\{x_1^{(0)}, g) + d(x_2^{(0)}, g) : g \in G\} + \varepsilon$

i.e. $d(x_1^{(0)}, g_0) + d(x_2^{(0)}, g_0) \leq \inf(d\{x_1^{(0)}, g) + d(x_2^{(0)}, g) : g \in G\} + \varepsilon$

i.e. $g_0 \in P_{G(\varepsilon)}(x_1^{(0)}, x_2^{(0)}) \subset P_{G(\varepsilon)}$(AxA). Hence $P_{G(\varepsilon)}$(AxA) is compact.

4.2 Best Simultaneous Co-Approximation

This section deals with the problem of best simultaneous co-approximation in metric linear spaces and convex metric spaces.

To start with, we recall a few definitions.

Definition 4.2.1. Let (X, d) be a metric space and G a non-empty subset of X. An element $g_0 \in g$ is called a best simultaneous co-approximation to x, y $\in$ X from G if

$$d(g_0, g) \le \max\{d(x, g)\}\ d(y, g) \text{ for all } g \in G.$$

The set of all best simultaneous co-approximations to x, y $\in$ X from G is denoted by $S_G(x,y)$. G is called an existence set if $S_G(x,y)$ contains at least one element, G is called a uniqueness set if $S_G(x,y)$ contains atomst one element and G is called an existence and uniqueness set if $S_G(x,y)$ contains exactly on element.

Definition 4.2.2. A metric linear space (x,d) is said to be totally complete [6] if it has the property that its d-bounded closed sets are compact.

Every totally complete metric linear space is finite dimensional but a finite-dimensional metric linear space need not be totally complete [6]. However, finite dimensional normed linear spaces are totally complete.

Some properties of the set $S_G(x,y)$ are as under:

Lemma 4.2.1. If G is a subset of a metric space (x,d) and x,y $\in$ X, then

(a) if $g_0 \in S_G(x,y)$ then for every $g \in G$

$$d(x, g_0) \le 2 \max\{d(x, g), d(y, g)\}$$

$$d(y, g_0) \le 2 \max\{d(x, g), d(y, g)\},$$

(b) $S_G(x,y)$ is bounded,

(c) $S_G(x,y)$ is closed if G is closed.

Proof.

(a) is easy to verity

(b) Let $g_0 \in S_G(x,y)$. Then by part (a), we have

$$d(x, g_0) \le 2 \max\{d(x, g), d(y, g)\}$$

for all $g \in G$ and so

$$d(g_0, x) \le 2 \inf \max_{g\in G} \{d(x, g), d(y, g)\}$$
$$\equiv 2\, d(x, y; G).$$

Then for arbitrary $g_0, g_0' \in S_G(x, y)$

$$d(g_0, g_0') \le d(g_0, x) + d(x, g_0')$$
$$\le 4d(x,y;G)$$

implying thereby that $S_G(x, y)$ is bounded.

(c) Let $\{g_n\}$ be any sequence of element of $S_G(x, y)$ such that $\{g_n\} \to g_0$.

Since G is closed, $g_0 \in G$. For any $g \in G$,

Consider

$$d(g_0, g) \le d(g_0, g_n) + d(g_n, g)$$
$$\le (g_0, g_n) + \max\{d(x, g), d(y, g)\}$$
$$\to 0 + \max\{d(x, g), d(y, g)\} \text{ as } n \to \infty.$$

Thus $g_0 \in S_G(x, y)$ and so $S_G(x, y)$ is closed.

For normed linear spaces Lemma 4.2.1 was proved in [13] – proposition 3.1.

Note : The following results can be easily verified :

(i) For $g_0 \in S_G(x, y)$, $\max d(g_0, x), d(g_0, y)\} \le 2\max\{d(x,g), d(y,g)\}$

(ii) For $g_0 \in P_G(x, y)$, $d(g_0, g) \le 2 \max\{d(x,g), d(y,g)\}$

(iii) For $g_0 \in P_G(x)$, $d(g_0, g) \le 2\, d(x,g)$

(iv) For $g_0 \in R_G(x)$, $d(g_0, x) \le 2d(x,g)$

for all $g \in G$.

(v) For $x \in X \backslash G$ and $g_0 \in G$ if g_0 is a best co-approximation to x from G, then g_0 is a best simultaneous co-approximation to x, y from G, for every $y \in X \backslash G$. But, if g_0 need not be a best co-approximation to either x or y. Therefore, $R_G(x) \cup R_G(y) \subset S_G(x, y)$.

It was proved in [1] that if G is a convex subset of a convex metric space (X,d), x, $y \in X$ and g_1, g_2 are best simultaneous approximations to x and y by the elements of G then $W(g_1, g_2, \lambda) \in G$ is also a best simultaneous approximation to x, y. The following theorem shows that the same is true in case of best simultaneous co-approximation:

Theorem 4.2.1. [14]. If G is a convex subset of a convex metric space (X, d) and x, $y \in X$. Then $S_G(x, y)$ is convex.

Proof. Let $g_1, g_2 \in S_G(x,y)$ and $\lambda \in [0,1]$.

Then $W(g_1, g_2, \lambda) \in G$ as $g_1, g_2 \in G$ and G is a convex set.

Consider

$$d(W(g_1, g_2, \lambda), g) \leq \lambda d(g_1, g) + (1-\lambda)\, d(g_2, g)$$

$$\leq \max\{d(g_1, g), d(g_2, g)\} \tag{4.2.1}$$

Since $g_1, g_2 \in S_G(x,y)$,

$$d(g_1, g) \leq \max\{d(x,g), d(y, g)\}, \text{ and} \tag{4.2.2}$$

$$d(g_2, g) \leq \max\{d(x,g), d(y, g)\}.$$

Now (4.2.1) and (4.2.2) imply

$$d(W(g_1,\ g_2,\lambda),g)\ \leq\ \max\ \{d(x,g)\ ,\ d\ (y,\ g)\}$$

for every $g \in G$ and so $W(g_1,\ g_2,\lambda)\in\ \ S_G\,(x,y)$.

It was proved by Diaz and McLaughlin [20] that for a normed linear space X, a finite dimensional subspace G of X and x, $y \in X/G$, if $g_0 \in G$ is a best approximation to (x_y)/2 from G, then g₀ is not a best simultaneous approximation to x, y from G. However, in case of best co-approximation we have:

Theorem 4.2.2 [14]. Let (X, d) be a convex metric space, G a subset of X, x, $y \in X/G$, $g_0 \in G$ and $0 \leq \alpha \leq 1$. If g_0 is a best co-approximation to W (x, y,α) for some α, then g_0 is a best simultaneous co-approximation to x, y from G.

Proof. Assume that g_0 is a best co-approximation to W (x, y,α) for some $\alpha \in [0,1]$. Then for every $g \in G$ it follows that

$$\begin{aligned} d\,(g_2,\ g) \quad &\leq\ d(W(x,y,\alpha),g) \\ &\leq\ \alpha\ d\,(x,g) + (1\text{-}\alpha)\ d\,(y,\ g) \\ &\leq\ \max\ \{d\,(x,\ g),\ d\,(y,\ g)\}. \end{aligned}$$

Thus g_0 is a best simultaneous co-approximation to x, y for G.

Remark. For normed linear spaces this result was proved in [10] – Theorem 4.4

For a convex metric linear space (X, d) and x, y belonging to convex set G in X, αx + (1-α)y, $0 \leq\alpha\leq 1$ is a best simultaneous co-approximation to x, y from G. As for any $g \in G$

$d(\alpha x + (1-\alpha) y, g) \le \alpha d(x,g) + (1-\alpha) d(y,g)$ (by the convexity of (X, d))

$$\le \max \{d(x, g), d(y, g)\}$$

Also every element belonging to $S_G(x, y)$ is of the form

$\alpha x + (1-\alpha) y, \quad 0 \le \alpha \le 1.$

Hence for a convex metric linear space (X, d) and a convex subset G of X, if $x, y \in G$ then

$$S_G(x,y) = \{\alpha x + (1-\alpha) y : 0 \le \alpha \le 1\}.$$

In the next theorem, we list some more properties of the set $S_G(x,y)$ in metric linear spaces.

Theorem 4.2.3 [14]. Let (X, d) be a metric linear space , G a subspace of X and $x, y \in X$. Then the following results hold:

(i) $S_G(x + g, y + g) = S_G(x, y) + g$ for $g \in G$.

(ii) $S_G(\alpha x, \alpha y) = \alpha S_G(x, y)$ for $x, y \in G$, $\alpha \in R$

The prof of Theorem 4.2.3 is a minor modification of that of Proposition 3.3 given in [10], for normed linear spaces.

The next result gives another property of the set $S_G(x,y)$ in totally complete metric linear spaces.

Theorem 4.2.3 [14]. Let (X, d) be a totally complete metric linear space , G a non-empty closed subset of X. Then $S_G(x,y)$ is compact.

Proof. Since $S_c(x, y)$ is closed and bounded (lemma 4.21) and the space is totally complete. S_G (x, y) is compact.

The following theorem proves the upper semi-continuity of the mapping S_G for totally complete metric linear spaces:

Theorem 4.2.5 [62] Let (X, d) be a totally complete metric linear space (X, d). Then the set-valued map $S_G\{x, y)$: $x, y \in X\}$--->$2^G$is upper semi-continuous.

Proof. Let A be a closed subset of G, we have to show that

$B=\{(x,y) : x, y \in X,\ S_G(x,y)\ \cap A \neq \phi\}$ is a closed subset of X. Let <(x_n, y_n)> be a sequence in B such that <(x_n, y_n)>---> (x_0, y_0) for some (x_0, y_0) $\in X$. Since $(x_n,y_n)\ \cap A$ is non-empty, choose

$g_n \in S_G(x_n,y_n)\ \cap A$ for each n.

Consider the set C=closure of the set {g_1, g_2, ..., g_n, ...}. Using Lemma 4.2.1 (a) , it is easy to see that the set C is a bounded set. Since C is a closed and bounded subset of the totaly complete space G, C is compact. Therefore the sequecne <g_n> has a subsequence $\{g_{n_k}\}$converging to g_0. Since A is closed, $g_0 \in A$.

Now to prove that B is closed, it is sufficient to prove that $g_0 \in S_G(x_0, y_0)$. For every $g \in G$.

$$\begin{aligned} d\,(g, g_0) &\leq d\,(g, g_{n_k}) + d\,(g_{n_k}, g_0) \\ &\leq \max\{d\,(x_{n_k}, g), d\,(y_{n_{pk}}, g)\} + d\,(g_{n_k}, g_0) \end{aligned}$$

$$\rightarrow \quad \max\{d(x_0, g), d(y_0, g) + 0 \text{ as } n \rightarrow \infty.$$

Therefore $g_0 \in S_G(x_0, y_0) \cap A$ and so $(x_0, y_0) \in B$ i.e. B is closed

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