MODULE* – 3

ON ε-BIRKHOFF ORTHOGONALITY

AND

ε-NEAR BEST APPROXIMATION

An element x of a normed linear space E is said to be orthogonal (in the sense of Birkhoff) to an element y є E if $\|x+\alpha y\| \geq \|x\|$ for every scalar α. The notion of Birkhoff orthogonality was used to prove some results on best approximation in normed linear spaces (see e.g. [6]). This notion of orthogonality was extended to metric linear spaces by T.D. Narnag [3] and some results on best approximation were proved. A generalization of Birkhoff Orthogonality, called ε-birkhoff orthogonality was introduced by Sever Silvestru Dragomir [1] in normed linear spaces ($\|x+\alpha y\|$) $\geq$ (1-ε)$\|x\|$ for all scalars α) and this notion was used to prove a decomposition theorem ([1]-Theorem 3). We extend the decomposition theorem in metric linear spaces (Theorem 3.1)

For a subset A of a normed linear space X and a +ve number ε, an ε-near best approximation of A by M is a map $\phi: A \rightarrow M$ such that $\|x-\phi(x)\| \leq d(x,M)+\varepsilon$ for all x in A. This notion of ε-near best approximation was used by Paul C. Kainen et al [2] to show that the existence of a continuous e-near best approximation in a strictly convex normed linear space X and taking values in a suitable subset M implies that M has the unique best approximation property. We extend this result to convex metric linear spaces (Theorem 3.2). we also extend some results on e-near best approximation proved in [1] to metric linear spaces (Theorem 3.3 and its corollaries).

To start with, we recall a few definitions.

Definition 3.1 [4]. Given a non-empty subset A of a metric space (X,d) and a positive number ε, near best approximation of A by M is a map ø:A→M such that

$$d(x, ø(x)) \leq d(x, M) + \varepsilon$$

for all x in A.

Definition 3.2 [1]. For a normed linear space X, over a field K (K=R or C) and $\varepsilon \in]0,1[$, an element XxX is said to be ε-Birkhoff Orthogoanl to y є X if

$$(\|x+\alpha y\|) \geq (1-\varepsilon)\|x\| \text{ for all } \alpha \in K$$

We denote it by $x \perp y$ (ε-B).

For a metric linear space (X,d) over field K and $\varepsilon \in]0,1[$, an element $x \in X$ is said to be ε–Birkhoff Orthogonal to $y \in X$ [1] if

$d(x+\alpha y, 0) \geq (1-\varepsilon)\, d(x, 0)$ for all $\alpha \in K$ and we denote it by

$x \perp y$ (ε-B).

If A is a non-empty subset of X then by ε–Birkhoff Orthogonal Complement $A^{\perp}(\varepsilon\text{-B})$, we denote the set of all elements of X which are ε –Birkhoff Orthogonal to A i.e.

$A^{\perp}(\varepsilon\text{-B})=\{ y\in X : y\perp x(\varepsilon\text{-B}) \text{ for all } x \in A\}$.

Since $A^{\perp}(\varepsilon\text{-B}) = \{ y\in X : y\perp x(\varepsilon\text{-B}) \text{ for all } x\in A\}$, $0\in A^{\perp}(\varepsilon\text{-B})$ as $0 \perp x$ (ε-B) for all $x\in A$ ($d(0+\alpha x\ 0) \geq (1-\varepsilon)(d(0,0)$ for all $x\in A$).

We claim that $A\cap A^{\perp}(\varepsilon\text{-B})\subseteq\{0\}$ for every $\varepsilon\in\ ]0, 1[$.

Let $y \in A\cap A^{\perp}(\varepsilon\text{-B})$. Then $y\in A^{\perp}(\varepsilon\text{-B})$.

Now $y\in A^{\perp}(\varepsilon\text{-B}) \Rightarrow y\perp x(\varepsilon\text{-B})$ for all $x\in A$.

$\Rightarrow y\perp y(\varepsilon\text{-B})$

$\Rightarrow d(y+\alpha y) \geq (1-\varepsilon)\, d(y,0)$ for all $\alpha\in K$

$\Rightarrow 0 \geq (1-\varepsilon)\, d(y,0)$ by taking $\alpha=-1$

$\Rightarrow d(y, 0) \leq 0$ (as $(1-\varepsilon)\geq 0$)

$\Rightarrow d(y, 0) = 0$

$\Rightarrow y = 0$

and so $A\cap A^{\perp}(\varepsilon\text{-B})\subseteq\{ 0 \}$.

we now prove a lemma needed in the proof of decomposition theorem.

Lemma 3.1 [5]. Let G be a closed linear subspace of a metric linear space (X,d), $G \neq X$. Then for any $\varepsilon\in\ ]0, 1[$, the ε–Birkhoff Orthogonal complement of G is non-zero.

Proof. Let $y \in X \backslash G$. Since G is closed, d (y, G) = r>0. Thus there exists $y_\varepsilon \in G$ such that

$$r \le d(y, y_\varepsilon) \le r/(1-\varepsilon) \quad (\text{as } r = d(y,G))$$

i.e. $r \le d(y-y_\varepsilon, 0) \le r/(1-\varepsilon)$.

Put $x_\varepsilon = y - y_\varepsilon$, we have $x_\varepsilon \neq 0$ and for all $y_1 \in G$ and $\lambda \in K$, we obtain

$$\begin{aligned} d(x_\varepsilon + \lambda y_1, 0) &= d(y - y_\varepsilon + \lambda y_1, 0) \\ &= d(y, y_\varepsilon - \lambda y_1) \\ &\ge r \ (\text{as } d(y,G) = r \text{ and } y_\varepsilon - \lambda y_1 \in G) \\ &\ge (1-\varepsilon)\, d(x_\varepsilon, 0) \end{aligned}$$

i.e. $x_\varepsilon \perp y_1(\varepsilon\text{-B})$ and so $x_\varepsilon \in G^\perp(\varepsilon\text{-B})$.

Using this lemma, we prove the following decomposition theorem in metric linear spaces (which for normed linear spaces was proved in [1])

Theorem 3.1 [5]. Let G be a closed linear subspace of a metric linear space (X,d). Then for any $\varepsilon \in]0,1[$, we have $X = G \oplus G^\perp(\varepsilon - B)$ **Proof.**

Suppose $G \neq X$ and $x \in X$. If $x \in G$, then $x = x + 0 \in G + G^\perp(\varepsilon - B)$.

If $x \notin G$, then there exists an element $y_\varepsilon \in G$ such that

$$0 < r = d(x, G) \le d(x, y_\varepsilon) \le r/(1-\varepsilon).$$

Since $x_\varepsilon = x - y_\varepsilon \in G^\perp(\varepsilon - B)$ (by the above lemma), we have

$x = y_\varepsilon + x_\varepsilon \in G + G^\perp(\varepsilon - B)$.

Since $\{0\} \subseteq G \cap G^\perp(\varepsilon - B) \subseteq \{0\}$, we get $X = G \oplus G^\perp(\varepsilon - B)$.

The following theorem shows that the continuity of ε-near best approximation is enough to guarantee the uniqueness of best approximation in convex in convex metric linear spaces which are pseudo strictly came x.

Theorem 3.2 [5]. Let (X, d) be a convex metric linear space which is pseudo strictly convex and M a boundedly compact closed subset of X. Suppose that for each $\varepsilon > 0$, there exists a contnuous ε-near best approximation ϕ:X---> M of X by M then M is a Chebyshev set.

Proof. Since a boundedly compact closed set in a metric space is proximinal (see [7], p. 283), $P_M(x)$ is non-empty for each $x \in X$. Let $m \in P_M(x)$

We choose a point $x_0 \in X$ with $r = d(x_0, M) > 0$. Given a +ve integer $n \geq 1$, let $\phi_n : X \to M$ be continuous with

$$d(x, \phi_n(x)) \leq d(x, M) + 1/n \text{ for all } x \text{ in } X.$$

Then $\phi_n : B(x_0, r) \to M$ and

$$d(\phi_n(x), x_0) \geq r \text{ for all } x \text{ in the closed ball } B(x_0, r).$$

Let π be a mapping defined by

$$\pi(x) = x_0 + r(x - x_0)/d(x, x_0),\ x \in X.$$

We claim that

$$\pi = \{x : d(x, x_0) \geq r\} \to \{x : d(x, x_0) = r\} \equiv \partial B(x_0, r)$$

is a redial retraction i.e.

(i) $d(\pi(x), x_0) = r$

(ii) for $x \in \partial B(x_0, r)$, $\pi(x) = x$.

Consider

$$d(\pi(x), x_0) = d(x_0 + r(x - x_0)/d(x, x_0), x_0)$$
$$= d(r(x - x_0)/d(x, x_0), 0),$$

$\geq$ r d $(x-x_0, 0)/d(x, x_0)$, by the convexity of (X , d)

$=$ rd $(x, x_0)/d(x, x_0)$

$=$ r .

Thus,

$$d(\pi(x), x_0), \leq r \qquad (3.1)$$

As $\pi(x) = x_0 + [r(x-x_0)]/d(x-x_0)$

$= r\,x/d(x, x_0) + [(1-r)/d(x-x_0)]\,x_0$

i.e. $\pi(x) \in [x, x_0]$ and so

$$d(x, \pi(x))) + d(\pi(x), x_0) = d(x, x_0) \qquad (3.2)$$

Now

$d(\pi(x), x) = d(x_0 + [r(x-x_0)]/d(x, x_0), x)$

$= d(r(x-x_0)/d(x, x_0), x-x_0)$

$\leq [1-r/d(x, x_0)]\,d(0, x-x_0)$, by the convexity of X

$= [1-r/d(x,x_0)]\,d(x,x_0)$

$= d(x,x_0) - r.$

Hence, $-d(\pi(x), x) \geq r - d(x,x_0)$, So (3.2) implies

$d(\pi(x), x_0) \geq d(x,x_0) + [r - d(x,x_0)] = r$

i.e. $d(\pi(x), x_0) \geq r$ (3.3)

Combining (3.1) and 3.3), we get $d(\pi(x, x_0) = r$.

For $x \in \partial B(x_0, r)$ i.e. $d(x, x_0) = r$, we get

$\pi(x) = x_0 + r(x-x_0)/d(x, x_0)$

$= x$

i.e. $\pi(x) = x$ for all $x \in \partial B(x_0, r)$.

Thus π : $\{ x : d(x, x_0) \geq r\} \to \{x : d(x, x_0) = r\}$ is a radial retraction and $\pi_0\phi_n : B(x_0, r) \to \partial B(x_0, r)$.

Now $\phi_n(x)$, for x in $B(x_0, r)$ satisfies.

$$d(\phi_n(x), x_0) \leq d(x, M) + 1/n + d(x, x_0) \qquad (3.4)$$
$$\leq d(x, x_0) + d(x_0, M) + 1/n + d(x, x_0)$$
$$= d(x_0, M) + 1/n + 2d(x, x_0)$$
$$\leq 3r + 1.$$

Hence $\phi_n(B(x_0, r)) \subseteq M \cap B(x_0, 3r+1)$ and $\phi_n(B(x_0, r))$ is a bounded subset of M. So c1 $(\phi_n(B(x_0, r))$ is compact since M is given to be boundedly compact.

Let P : X $\to$ X be the reflection through x_0.

i.e. $P(y) = x_0 + (x_0 - y)$ (3.5)

Then cl $(P_0\pi_0\phi_n(B(x_n(B(x_0, r))) = P_0\pi(\text{cl}\phi_n(B(x_0, r)))$ is compact subset of $\partial B[x_0, r]$ and $P_0\pi_0\phi_n$ is a continuous function from $B(x_0, r)$ into $\partial B(x_0, r)$.

Since in a convex metric linear space $B(x_0, r)$ is convex, by Rothe's theorem, a version of Schauder's theorem (see [74], p. 27) for each n, $P_0\pi_0\phi_n$ has a fixed point x_n in $B(x_0, r)$.

Thus $x_n = P_0\pi_0\phi_n(x_n)$

$= P_0(\pi_0\phi_n(x_n))$

$= 2x_0 - (\pi_0\phi_n(x_n))$ (using (3.5))

and so $(\pi_0\phi_n)(x_n) = 2x_0 - x_n$

We claim that x_n, x_0, $2x_0 - x_n = \pi_0\phi_n(x_n)$ and $\phi_n(x_n)$ are consecutive collinear points.

Since $2x_0 - x_n = \pi_0\phi_n(x_n)$ implies $2x_0 - x_n - \pi_0\phi_n(x_n) = 0$ i.e.

$\alpha x_0 + \beta x_n + \gamma\pi_0\phi_n(x_n) = 0$ with $\alpha+\beta+\gamma=0$ i.e. $x_0 + \beta x_n + \gamma.\pi_0\phi_n(x_n)/(\beta+\gamma)$.

Also, by the definition of π (x) we have

$$\pi(\phi_n(x_n)) = x_0 + (r(\phi_n(x_n) - x_0))/\ d(\phi_n(x_n), x_0)$$

$$= r\phi_n(x_n)/d(\phi_n(x_n), x_0) + (1 - r/[d(\phi_n(x_n), x_0])x_0$$

$$\Rightarrow 1.\pi_0\phi_n(x_n) - r\phi_n(x_n)/d(\phi_n(x_n), x_0) - (1 - r/d(\phi_n(x_n), x_0))x_0 = 0$$

$$\Rightarrow \alpha.\pi_0\phi_n(x_n) + \beta\phi_n(x_n) + \gamma.x_0 = 0$$

with $\alpha+\beta+\gamma = 1 - r/d(\phi_n(x_n), x_0) - 1 + r/d(\phi_n(x_n), x_0) = 0$

$$\Rightarrow \pi(\phi_n(x_n) = (\beta\phi_n(x_n) + \gamma.x_0)/(\beta+\gamma)$$

and so

$$d\ (\phi_n(x_n), x_n) \geq d(\pi_0\phi_n(x_n), x_n)$$

$$= d(2x_0 - x_n, x_n)$$

$$= d(x_n, x_0) + d(x_0, 2x_0 - x_n),\ as\ x_n, x_0$$

and $2x_0 - x_n$ are collinear

$$= d(x_n, x_0) + d(x_n, x_0)$$

$$= 2d(x_n, x_0)$$

Now we prove that $d(x_n, x_0) = r$

Since $\pi_0\phi_n : B(x_0, r) \to \partial B(x_0, r)$ and $x_n \in B(x_0, r)$

implies $(\pi_0\phi_n\ (x_n)x_0) = r$ i.e $d\ (2x_0 - x_n, x_0) = r$, i.e. $d(x_n, x_0) = r$.

Hence d $(\phi_n\ (x_n),\ x_n) \geq 2r$.

In addition for each m in M,

$d(x_n, m) \geq d(x_n, \phi_n(x_n)) - 1/n$ (using (3.4))

$$\geq 2r - 1/n \qquad (3.6)$$

Again M is boundedly compact, the sequence $\{\phi_n(x_n)\}$ **in** $M \cap B(x_0, 3r+1)$ **has a convergent subsequent with limit u in x. Then the sequence** $\{P_o\pi_0\phi_n(x_n)\}$ **has a convergent subsequence with limit** $P_o\pi(u) = x_\infty \in \partial B(x_0, r)$ **.**

Moreover, for each m in M,

$$d((x_\infty - x_0) + (x_0 - m), 0) = d((x_\infty - m, 0)$$

$$= d(x_\infty,\ m)$$

$$\geq 2r(\text{using} \quad (3.6)) \qquad (3.7)$$

If m is in $P_M(x_o)$ *then* $d\,(x_0, m) = d(x_0, m) = r.$

Also $d(x_\infty, x_0) = r$ as $x_\infty \in \partial B\ (x_0,\ r)$. So

$$d((x_\infty - x_0) + (x_0 - m), 0) \ = d(x_\infty - x_0, m - x_0)$$

$$\leq d(x_\infty - x_0, 0) + d(m - x_0, 0)$$

= r + r

= 2r

Implies

$$d((x_\infty - x_0) + d(x_0 - m), 0) \leq 2r \quad (3.8)$$

Combining (3.7) and (3.8) we have

$$d((x_\infty - x_0) + d(x_0 - m), 0) = 2r$$

= r + r

$$= d((x_\infty - x_0), 0) + d((x_0 - m), 0) \qquad (3.9)$$

Since (X, d) is pseudo strictly Convex, (3.9) implies

$x_\infty - x_{0,} = t(x_0 - m)$ for some $t > 0$.

i.e. m =[(1+t) x_0-x_∞]/t implying $P_M(x_0)$=[(1+t) x_0-x_∞]/t for t>0. Hence M is Chebyshev.

In strictly convex normed linear spaces this theorem was proved by Paul C. Kainen et al [2] and the above proof is an extension of the one given in[2].

Corollary 3.1 [5]. Let (X, d) be a convex metric linear space, M a boundedly compact subset of X and x an element of X with r=d(x_0, M)>0. Suppose that for some ε, with 0<ε<2r there exists continuous ε-near best approximation $\phi : B(x,r) \to M$ of B (x, r) by M. Then there exists a point x_1 in $\partial B(x,r)$ such that d (x_1, m) $\geq$2r-ε.

Proof. The proof of this is contained in the first part of the proof of Theorem 3.2 (upto equation (3.6)).

If M is an approximatively compact set in a metric space, then $P_M(x)$ is compact for each x in X. Indeed, any $\{m_n\}$ in $P_M(x)$ is a sequence in M with d(x, m_n) = d(x, M) and by the definition of approximative compactness, has a convergent subsequence with limit in M and hence in $P_M(x)$. Using this we have:

Theorem 3.3 [5]. Let M be an approximatively compact set in a metric linear space (X, d) and x an element of X. Suppose that for each ε>0, there is a continuous ε-near best approximation $\phi_\varepsilon : (x) \cup P_M(x) \longrightarrow M$.

Then $P_M(x)$ is connected.

For normed linear spaces the proof of Theorem 3.3 is given in [2] and that proof can easily be extended to metric linear spaces.

Corollary 3.2 [5].Let (X, d) be a metric linear space and M an approximately (i.e.. $P_M(x)$ is non-empty and countable for each x in X). Suppose that for each $\varepsilon > 0$ there exists a continuous ε-near best approximation $\phi : \mathrm{x} \to \mathrm{M}$ of X by M. Then M is a Chebyshev set.

Proof. By Theorem 3.3 for each x, $P_M(x)$ is connected and since the only countable connected set is a singleton, M is Chebyshev,

Corollary 3.3 [5]. Let (X, d) be a metric linear space, M a closed, boundary compact subset of X, and x an element of X with r = d (x, M) > 0. If for each $\varepsilon > 0$, there exists a continuous ε nearbest approximation $\phi : B(x, r) \to M$ of B(x ,r) by M then $P_M(x)$ is connected.

Proof Since a closed , boundedly compact subset is approximatively compact ([6],p. 383), proof follows from Theorem 3.3 .

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