MODULE* – 2

ON BEST APPROXIAMTION AND METRIC PROJECTIONS

For a given point x and a given set G of a metric space (X,d), a point g_0ϵG satisfying d (x,g)=inf {d(x,g):geG} is called a best approximation to x in G and the map which takes each point xeX to set of its best approximation map or the metric projection of X onto G. This module dealing with best approximation and metric projections, has been divided into two sections. First section is concerned with best approximation in pseudo strictly convex metric linear spaces and second section with multi-valued metric projections in convex spaces.

The notion of pseudo strict convexity in metric linear spaces was introduced and discussed by K.P.R. Sastry and S.V.R. Naidu in [6] and [7]. It was shown by Paul C.Kainen et al [4] that the existence of a continuous best approximation in a strictly convex normed linear space X and taking values in a suitable subset M of X implies that M has the unique best approximation property. In the first section of this module, we extend this result of Paul C.Kainen et al to pseudo strictly convex metric linear spaces.

S.B.Stockin [14] provod that if U_M= {xcX: Card P_M (x) ≤ 1} then U_M =x for every subset M of X iff X is a strictly convex normed linear space.

This result was extended to strictly convex metric spaces by T.D. Narang [5]. A question that arises is what happens in spaces which are not strictly convex? To answer this, we have discussed the characterization of multi-valued metric projection P_M in spaces which are not strictly convex in the second section. For normed linear spaces this result was proved by Ioan Serb in [8]. We have also proved that for a non-void proper subset M of a complete convex metric linear space (X,d) P_M cannot be a countably multi-valued metric projection. We have given a characterization of the semi-metric linear spaces in terms of finitely multi-valued metric projections. In [9], it was proved that if M is a strongly proximinal subset of a Banach space X, then Card $P_M(x) \geq c$ for every $x \in X \setminus M$, and the completeness of the space is essential for the validity of the result. In [10], the same result was proved for complete metrizable locally convex spaces i.e. in Frechet spaces. In this section, we have proved that for a strongly proximinal set M in a complete convex metric space (X,d) , Cad $P_M(x) \geq$ c for all $x \in X \setminus M$.

2.1 Best Approximation

This section deals with best approximation in pseudo strictly convex metric linear spaces. To begin with , we recall a few definitions .

Definition 2.1.1 [15] A metric linear space (X,d) is said to be convex if for x, y $\in$X , $\lambda \in [0,1]$

$$d(u,\lambda x+(1-\lambda)y)\leq \lambda d(u,x)+(1-\lambda)d(u,y)$$

for all u $\in$X.

Definition 2.1.2 [11] A metric linear space (X,d) is said to be pseudo strictly convex (P.S.C.) if given $x \neq 0$, $y \neq 0$, $d(x+y,0)=d(x,0)+d(y,0)$ implies $y=tx$ for some $t>0$.

Strict convexity and pseudo strictly convexity are equivalent in normed linear spaces (see e.g. [1] p. 122 or [7])but not in metric linear spaces[7]. (For strict convexity in normed linear spaces one may refer to [3]).

The following example shows that a P.S.C. metric linear space need not be S.C.

Example 2.1.1 [7]

Let $f: R^+ \to R^+$ be defined by

$$f(t) = \begin{cases} t \text{ if } 0 \leq t \leq 1 \\ 1 \text{ if } t \geq 1 \end{cases}$$

and d be the metric on R defined by $d(0,t)= f(|t|)$ for all $t \in R$. Then (R,d) is P.S.C , but not S.C.

Definition 2.1.3 Let (X,d) be a metric space and M a subset of X. Given a non-empty subset A of X, a best approximation of A by M is a function $\phi: A \text{---}> M$ such that $d(x,\phi(x))=d(x,M)$ for x I n A.

The following theorem deals with the uniqueness of best approximation:

Theorem 2.1.1 [12] Let (X,d) be a convex metric linear space with pseudo strict convexity and let M a subset of X, let $\phi: X \text{---}> M$ be a continous best approximation of X by M. Then M is a Chebyshev set.

Proof : Since $\phi: X \text{---}> M$ is best approximation of X by M, $d(x,\phi(x))=d(x,M)$ for all $x \in X$ i.e. $P_M(x)$ is non-empty for each $x \in X$.

Now we show that $P_M(x)$ is a singleton . For any x in X, let $m \in P_M(x)$.

Suppose y is on the line segment [m,x) and $u \in P_M(y)$. Then

$$d(u,x) \leq d(u, y) + d(y, x)$$
$$\leq d(m, y) + d(y, x)$$
$$= d(m,x) \qquad \text{(as } y \in [m,x) \text{)}$$
$$\leq d(u,x) \text{ as } m \in P_M(x).$$

Therefore , the inequalities are all equalities and so $d(x,u)=d(x,m)=d(x,M)$ i.e. $u \in P_M(x)$ and therefore $P_M(y) \subseteq P_M(x)$.

Also m $\in P_M(y)$ as d(y ,u)+ d(y ,x)= d(m ,y)+ d(y ,x) implies d(y ,u)= d(y ,m).

Since d(u,x)=d(u,y)+d(y,x) i.e. d(u-x,0) = d(u-y,0)+ d(y-x,0), a consequence of pseudo strict convexity is that u, y and x are collinear. (By P.S.C. u-y= t(y-x) i.e. $y=u/(1+t)+tx/(1+t)$ and therefore u=m as u ,y , x are collinear and d(y,u)=d(y,m) . Hence $P_M(y)$ = {m}. Since ϕ is directionally continuous at x and $\phi([m,x\})=\{m\}$, it follows that $\phi(x)=m$. Thus $P_M(y)$ = $\{\phi(x)\}$ is a singleton set i.e M is a Chebyshev set.

Note : The following result is established in the proof of Theorem 2.1.1 without the requirement of pseudo strict convexity.

Let (X,d) be a convex metric linear space , M a subset of X,x an element of X and m an element of $P_M(x)$. Then for each y $\in$[m,x) , {m} $\subseteq P_M(y) \subseteq P_M(x)$.

Definition 2.2.1 Let (X,d) be a metric space , M a subset of X. The mapping $P_M : X ---> M$ defined by

$$P_M(x)=\{m \in M : d(x,m)=d(x,M)\}$$

Is called the multi – valued metric projection of X onto M.

By U_M, we denote the set

U_M = { x $\in$X: card $P_M(x) \leq 1$ } .

If card $P_M(x) \geq 2$, we say that the metric projection is totally multi-valued and if $2 \leq$ card $P_M(x) < \infty$, then the metric projection is called finitely multi-valued .In the special case when card $P_M(x)$ = χ_0, we say that the metric projection is countably multi-valued.

Definition 2.2.2 If P_M is a totally multivalued metric project then M is strongly proximinal.

It is clear that every strong proximinal is proximinal and hence closed.

(Let M be not closed. Let $x \in \overline{M}$ /M . Then d(x,M)=0. Since M is proximinal , there exists m $\in$M such that d(x, m) =d(x, M)=0 and so x=m i.e. x $\in$M , a contradiction.)

It was shown in [5] that if (x,d) Is a strictly convex metric space, then the corresponding P_M a single valued metric projection. A question that arises is what happens in spaces which are not strictly convex. We have:

<u>Theorem 2.2.1</u> [13] Let M be an arbitrary non-void proper subset of a convex metric space (X, d) .Then P_M is not a finitely multivalued metric projection .

Proof <u>Case 1</u> If $\overline{M}$ = X then for every x $\in$X\M , we have $P_M(x) = \phi$ and hence P_M is not a finitely multivalued metric projection .

<u>Case 11</u> If $\overline{M} \neq$X, let $x_0 \in$X\ $\overline{M}$ then there exists a neighbourhood of x_0 contained in X\ $\overline{M}$.

Let r= d(x_0 , M) >0 . Suppose P_M is a finitely multivalued metric projection.

Let $P_M(x_0) = \{m_1, m_2, ..., m_k, k \geq 2\}$.

Then $d(x_0, m_i) = r$, I = 1,2,…,k.

Let $y_0 = W(x_{0,} m_i, \lambda)$, 0< λ <1 and 0<3 λ r < $\min\limits_{2 \leq i \leq k} d(m_1, m_i)$.

We claim that

(1) $B(y_0, \lambda r) \subseteq B(x_0, r)$

(2) $B(y_0, \lambda r) \cap M = \{m_1\}$

(1) Let $x \in B(y_0, \lambda r)$ then

$$d(x, x_0) \leq d(x, y_0) + d(y_0, x_0)$$

$= \ d(x, y_0) + d(W(x_0, m_1, \lambda), x_0)$

$\le d(x, y_0) + \lambda d(x_0, x_0) + (1-\lambda) d(m_1, x_0)$

$\le \lambda r + \lambda.0 + (1-\lambda) r$

$=$ **r .**

This implies that $x \in B(x_0, r)$. **Hence** $B(y_0, \lambda r) \subseteq B(x_0, r)$.

(2) **Since** $B(y_0, \lambda r) \subseteq B(x_0, r)$ **and** $B(x_0, r) \cap M = \{m_1, m_2, \ldots, m_k\}$**, it follows that** $B(y_0, \lambda r) \cap M = \{m_1, m_2, \ldots, m_k\}$.

We first show that

(a) $m_1 \in B(y_0, \lambda r)$

(b) $m_i \notin B(y_0, \lambda r), i = 1, 2, \ldots, k$

(a) **Consider**

$d(y_0, m_1) = d(W(x_0, m_1, \lambda), m_1)$

$\le \lambda d(x_0, m_1) + (1-\lambda) d(m_1, m_1)$

$= \lambda$ **r**

Which implies that $m_1 \in B(y_0, \lambda r)$

(b) **Since**

$d(m_1, m_i) \le d(m_1, y_0) + d(y_0, m_i)$

$d(y_{0,} m_i) \ge d(m_1, m_i) - d(m_1, y_0)$

$\ge 3\lambda r - \lambda r$

$= 2 \lambda$ **r**

$> \lambda$ **r**

So $m_i \notin B(y_0, \lambda r), i = 1, 2, \ldots, k$

From (a) and (b), it follows that $B(y_0, \lambda r) \cap M = \{m_1\}$.

We claim that $y_0 \in U_M$ .Consider

$$d(y_0,m_1)=d(W(x_0,m_1,\lambda),m_1)$$

$$\le \lambda d(x_0,m_1)+(1-\lambda)d(m_1,m_1)$$

$$= \lambda \text{r.}$$

Thus $d(y_0,m_1)\le \lambda r$ and so $d(y_0,M)\le \lambda r$ (2.2.1)

Now let $d(y_0,M)\le \lambda r$ .Then $d(y_0,m)=d(y_0,M)\le \lambda r$ (using (2.2.1)) $\Rightarrow m\in B(y_0,\lambda r)\Rightarrow\Rightarrow m\in B(y_0,\lambda r)\cap M=\{m_1\}$.So $P_M(y_0)\subseteq\{m_1\}$ and hence $card P_M(y_0)\le 1$ which implies that $y_0\in U_M$ .Thus P_M is not a finitely multi-valued metric projrction contradicting the hypothesis.

<u>Remark 2.1.1</u>For normed linear spaces this result was proved in [8]:

It may be remarked that for a convex set M in a convex metric space the corresponding P_M isn't a finitely multi-valued metric projection.

Indeed, if M is a convex set, $x\in X\setminus M$ and if $m_1,m_2\in P_M(x)$ with $m_1\ne m_2$, then for every $\lambda\in(0,1)$ we have

$$d(x,W(m_1,m_2,\lambda))\le \lambda d(x,m_1)+(1-\lambda)d(x,m_2)$$

$$=\lambda d(x,M)+(1-\lambda)d(x,M)$$

$$=d(x,M)$$

Also by the definition of $d(x,M)$ we have $d(x,M)\le d(x,W(m_1,m_2,\lambda))$ as $W(m_1,m_2,\lambda)\in M$ by the convexity of M.

Hence $d(x,W(m_1,m_2,\lambda)=d(x,M)$ i.e. $W(m_1,m_2,\lambda)\in M$ for every $\lambda\in(0,1)$

The. following result on metric projection was proved by Ioan Serb [58].

Let M be a non-void proper subset of a metrizable vector space X. If P_M is a countable multi-valued metric projection, then P_M is a perfect subset of X. Using this we have:

__Theorem 2.2.2__[13]If (X,d) is a complete metric linear space and M a non-void proper subset of a X then P_M cannot be a countably multi-valued metric projection.

Proof We suppose that there exists a set $M \subseteq X$ with the property that P_M is a countably multi-valued metric projection. So by the above result of Ioan Serb, M is a perfect set. If $x \in X \setminus M$ then $card P_M(x) = \chi_0$. We claim that $P_M(x) = \varepsilon$, where $S(x, d(x, M))$ is the sphere with centre x and radius d(x , M) , is a perfect set.

(a) $P_M(x)$ is a closed set as it is an intersection of two closed sets.

(b) $P_M(x)$ is dense in itself.

Indeed, if m_0 is an isolated point of $P_M(x)$, then there exists a ball $B(m_0, \varepsilon)$ with centre m_0 and radius $\varepsilon > 0$ such that $P_M(x) \cap B(m_0, \varepsilon) = \{m_0\}$.

Let us consider the point $x_\lambda = W(x, m_0, \lambda)$ with $0 < \lambda < \varepsilon \setminus [2d(x, m_0)] < 1$ since

$$d(x, x_\lambda) = d(x, W(x, m_0, \lambda))$$

$$\leq \lambda d(x, x) + (1-\lambda) d(x, m_0)$$

$$= (1-\lambda) d(x, m_0)$$

$$= (1-\lambda) d(x, M) \qquad \textbf{(2.2.2)}$$

It follows that $x_\lambda \in X \setminus M$.

Now

$$d(x_\lambda, m_0) = d(W(x, m_0, \lambda), m_0)$$

$$= \lambda d(x, m_0) + (1-\lambda) d(m_0, m_0)$$

$$= \lambda d(x, m_0) \qquad \textbf{(2.2.3)}$$

$$< \varepsilon \setminus 2.$$

On the other hand, let $m \in M, m \neq m_0$. Then $m \in P_M(x) \setminus \{m_0\}$ or $m \notin B(x, d(x, M))$. We shall prove that $d(x, m_0) > d(x_\lambda, m_0)$ in both the cases.

If $m \in P_M(x) \setminus \{m_0\}$, we have $d(x_\lambda, m) \geq |d(m_0, m) - d(x_\lambda, m_0)|$.

The proof of which is as under :

$$d(x_\lambda, m_0) = d((x_\lambda - m) + (m - m_0), 0)$$

$$\leq d((x_\lambda - m), 0) + d((m - m_0), 0)$$

$$\Rightarrow d(x_\lambda, m) \geq d(x_\lambda, m_0) - d(m, m_0) \qquad \textbf{(2.2.4)}$$

and

$$d(m_0, m) = d((m_0 - x_\lambda) + (x_\lambda - m), 0)$$

$$\leq d(m_0, x_\lambda) + d(x_\lambda, m)$$

$$\Rightarrow d(x_\lambda, m) \geq d(m, m_0) - d(m_0, x_\lambda) \quad \textbf{(2.2.5)}$$

Combining (2.2.4) and (2.2.5) , we get the result.

So $d(x_\lambda, m) \geq |d(m_0, m) - d(x_\lambda, m_0)|$

$> \varepsilon - \varepsilon/2$ (Since $m \notin \{m_0\} = P_M(x) \cap B(m_0, \varepsilon)$ and $m \in P_M(x)$)

$= \varepsilon/2$

$> d(x_\lambda, m_0).$

If $m \notin B(x, d(x, M))$ we have

$d(x_\lambda, m) \geq |d(x, m) - d(x, x_\lambda)|$

$> \lambda d(x, M) - (1-\lambda) d(x, M)$ (as (2.2.2) $\Rightarrow d(x, x_\lambda) \leq (1-\lambda) d(x, M)$)

$= \lambda d(x, M)$

$\geq d(x_\lambda, m_0)$ (as (4.2.3) $\Rightarrow d(x_\lambda, m_0) \leq d(x, m_0) = \lambda d(x, M)$).

Hence in both the cases we get that $d(x_\lambda, m) > d(x_\lambda, m_0)$. It follows that $P_M(x_\lambda) = \{m_0\}$ and $x_\lambda \in X \setminus M$. So P_M is not countably multi-valued, a contradiction. Therefore $P_M(x)$ has no isolated point. Thus $P_M(x)$ is dense in itself and hence a perfect set .But if $P_M(x)$ is a perfect set of a complete metric space X , then $P_M(x)$ is an uncountable set (see [2],p.72), contradicting our supposition. The theorem is thus proved.

<u>Remark 2.2.2</u> For Banach space this result was proved by Ioan Serb[8].

Next we shall give a characterization of the semi-metric linear spaces which aren't metric linear spaces in terms of finitely multivalued metric projections.

<u>Theorem 2.2.3</u>[13] . In every semi-metric linear space X, which isn't a metric linear space, there exist sets $M_2, M_3, \ldots, M_n, \ldots$ as well as the sets A and B such that

(1) $cardP_{M_n}(x)$ =n , for every $x \in X \backslash M_n$ and every $n \in N$, and

(11) $cardP_A(x) = \chi_0$ for every $x \in X \backslash A$,

(111) $cardP_B(x) = c$ for every $x \in X \backslash B$.

<u>**Proof**</u> **. Since X is a semi-metric but not a metric linear space, there exists an**

element $x_0 \neq 0$ **with** $d(x_0, 0) = 0$**.We shall prove that the sets**

$M_2 = \{x_0, x_0 \backslash 2\}$,

$M_3 = \{x_0, x_0 \backslash 2, x_0 \backslash 3\}$,

$M_n = \{x_0, x_0 \backslash 2, x_0 \backslash 3, ..., x_0 \backslash n\}$,

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$A = \{x_0, x_0 \backslash 2, x_0 \backslash 3, ..., x_0 \backslash n, ...\}$

and $B = \{\lambda x_0\}_{\lambda \in (0,1)}$ **have the properties** (1)----(111) **respectively.**

(1) **To prove** $CardP_{M_n}(x)$ **=n , for every** $x \in X \backslash M_n$ **and every** $n \in N$**.**

Let $x \in X \backslash M_n$**, then we have**

d(x , m)= d(x-m ,0)

≤ d(x ,0)+ d(-m ,0)

= d(x ,0)+ d(m ,0) **(2.2.6)**

We claim that d(m,0)=0 for all $m \in M_n$**.**

$m \in M_n \Rightarrow m = x_0 \backslash n$ **for same natural number n.**

Now for $\lambda \in (0,1)$ **,We have**

$d(\lambda x_0,0)=d(\lambda x_0+(1-\lambda)0,0)$

$\le \lambda d(x_0,0)+(1-\lambda)d(0,0)$

$=\lambda d(x_0,0)$.

For $\lambda=1\backslash n$, **We get** $d((1\backslash n)x_0,0)\le(1\backslash n)d(x_0,0)=0$.

Also $d(x_0\backslash n,0)\ge 0$ **implies that** $d(x_0\backslash n,0)=0$ **i.e. d(m,0)=0 for all** $m\in M_n$.**So ,** **(2.2.6) implies** $d(x,m)\le d(x,0)$.

Also $d(x,0)=d(x-m+m,0)$

$\le d(x-m,0)+d(m,0)$

$=d(x,m)$.

Hence, we have $d(x,0)=d(x,m)$ **for every** $m\in M_n$ **and so** $cardP_{M_n}(x)=cardM_n=n.$

(11) **As discussed above for any** $m\in A$, **we have** $m=x_0\backslash n, n\in N$ **and** $d(x,0)=d(x,m)$ **for all** $m\in A$ **which, further implies that** $cardP_A(x)=cardA$.

Since A is infinite countable set, by definition of χ_0, **we have** $cardP_A(x)=cardA=\chi_0$.

(111) **For** $m\in B$ **we have** $m=\lambda x_0$ **for** $0<\lambda<1$. **Again as discussed in** (1) **we have** $d(x,0)=d(x,m)$ **for all** $m\in B$ **which implies** $cardP_B(x)=B$. **Since B is an infinite uncountable set,** $cardP_B(x)=cardB.$

<u>Corollary 2.2.1</u>. The convex semi-metric linear space (X,d) isn't a metric linear space if and only if there exists a subset M of X such that P_M **is a finitely multi-valued metric projection.**

<u>Proof</u>Since X is a convex semi-metric linear space but not a metric linear space, there exists an element $x_0\ne 0$ **with** $d(x_0,0)=0$. **Then as discussed in**

Theorem 4.2.3, there exists a set $M = M_n = \{x_0, x_0 \backslash 2, x_0 \backslash 3, ..., x_0 \backslash n\}$ with the property that $cardP_{M_n}(x) = cardM_n = n$ i.e. P_M is a finitely multi-valued metric projection.

Conversly, suppose that there exists a subset M of X such that P_M is a finitely multi-valued metric projection. We are to prove that X isn't a metric linear space.

On contrary, if we assume X to be a metric linear space, then we get from Theorem 2.2.1 that P_M isn't a finitely multi-valued metric projection, a contradiction to the hypothesis.

Remark 2.2.3For semi-normed spaces the above result was proved in [8].

Considering M to be strongly proximinal subset of a Banach space X, Ioan Serb [9] proved that $cardP_B(x) \geq c$ for every $x \in X \backslash M$, and the completeness of the space is essential for the validity of the result. In [10], the same result was proved for complete metrizable locally convex spaces i.e. for Frachet spaces.

In convex metric spaces we have:

Theorem 2.2.4[12]. If M is strongly proximinal set in a complete convex metric space (X, d) , then $cardP_B(x) \geq c$ for all $x \in X \backslash M$,

Proof. Since M is strongly proximinal set, M is closed and so is $P_M(x) = M \cap B(x, d(x, M))$. We shall show that if $x \in X \backslash M$, $P_M(x)$ does not contain isolated points.

Suppose $m_0 \in P_M(x)$ is an isolated point of $P_M(x)$ for a given $x \in X \backslash M$ .Then there exists an $\varepsilon \in (0.1)$ such that $B(m_0, \varepsilon d(x, M)) \cap P_M(x) = \{m_0\}$.

(2.2.7)

Let $x_0 = W(x, m_0, \varepsilon \backslash 3)$, we have

$d(x, x_0) = d(x, W(x, m_0, \varepsilon \backslash 3))$

$\leq (1-\varepsilon\backslash 3)d(x,m_0)$ **(by the convexity of X)**

$= (1-\varepsilon\backslash 3)d(x,M)$

$< d(x,M)$,

It follows that $x_0 \in X\setminus M$ **. On the other hand**

$d(x_0,m_0) = d(W(x,m_0,\varepsilon\backslash 3),m_0)$

$\leq \varepsilon\backslash 3d(x,m_0)$ **(by the convexity of X)**

$= (\varepsilon\backslash 3)d(x,M)$. **(2.2.8)**

This implies

$d(x_0,m) \leq (\varepsilon\backslash 3)d(x,M)$. **(2.2.9)**

Let $m \in M$.**If** $m \notin P_M(x)$ **we have**

$d(x_0,m) \geq d(x,m) - d(x_0,x)$

$> d(x,M) - d(x,x_0)$

$\geq d(x,M) - (1-\varepsilon\backslash 3)d(x,M)$

$= (\varepsilon\backslash 3)d(x,M)$

i.e. $d(x_0,m) > (\varepsilon\backslash 3)d(x,M)$,

so $m \notin P_M(x_0)$(**if** $m \in P_M(x_0)$ **then** $d(x_0,m) \geq d(m_0,m) - d(m_0,x_0) \leq (\varepsilon\backslash 3)d(x,M)$ **by (2.2.9))**

If $m \in P_M(x)\setminus\{m_0\}$**, we have**

$d(x_0,m) \geq d(m_0,m) - d(m_0,x)$

$> \varepsilon d(x,M) - (\varepsilon\backslash 3)d(x,M)$ **(by (2.2.7) and (2.2.8))**

$= 2(\varepsilon\backslash 3)d(x,M)$

and so $m \notin P_M(x_0)$.

Thus for all $m \neq m_0, m \in M$, we have $m \notin P_M(x_0)$ and it follows that $P_M(x) = \{m_0\}$ and this contradicts the fact that M is a strongly proximinal set.

Thus $P_M(x)$ is a closed set and has no isolated point i.e. $P_M(x)$ is a perfect set in X for all $x \in X \setminus M$ .Since every perfect subset of a complete metric space has the cardinality at least c (Theorem 6.65 , p.72 [2]), $card P_B(x) \geq c$ for all $x \in X \setminus M$,

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